MATHSCI Problems

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1. Suppose x_1 and x_2 are the roots of

$$x^2 + x - 7 = 0.$$

Without solving for the roots, find

(a) $x_1^2 + x_2^2$.
(b) $x_1^3 + x_2^3$.
(c) $x_1^4 + x_2^4$.

Hint. It's possible to find sums and products of the roots without finding the roots themselves.

2. What is the difference between boiling and evaporation? Think about the similarities and differences. Focus on the essential features of both. Make a distinction between what is essential and what is not essential.

3. There are three fundamental circuit elements: resistor, capacitor and inductor. Why are they considered to be fundamental? Can we design other circuit elements that are not resistors, capacitors or inductors and build circuits out of them? Can there be other kinds of fundamental circuit elements or are these three unique? Why?

Comment. I don't expect an immediate answer to this. It may take weeks, months or years of thinking. Just keep this question in mind as you learn about electricity and electric circuits. Take your time thinking about this.

4. An orchestra is about to begin playing Beethoven's Fifth Symphony. There is a live audience, and the performance is also being broadcast over the radio. Your friend is $8,000$ kilometers away, listening on the radio. You are in the audience. How far must you sit from the orchestra so that you and your friend hear the opening notes at exactly the same time?

5. If b and c are odd, prove that

$$x^2 + bx + c = 0$$

cannot have two integer solutions.

6. Prove that $n^4 + 4$ is never prime, for all natural numbers $n > 1$.

Comment. A number m is prime if it has no factors except 1 and m. A number m is composite (not prime) if it has a factor other than 1 or m. So this problem asks you to show that $m = n^4 + 4$ is always composite. You can do this by showing that m has a factor that is not 1 and not m.

7. Find the radius and center of a circle passing through point $A = (1, 2)$ and tangent to the coordinate axes. Write the equation of the circle and draw it.

Comment. This is the problem that got me interested in doing more problems.

8. Which gravitational force is stronger, the Earth's attraction on the Moon, or the Sun's? Support your answer with calculations.

9. Consider the Earth and Moon and imagine a line going through the centers of both the Earth and the Moon. Call this the center line. There must be some point on this line where the Earth's gravitational attraction and the Moon's gravitational attraction are exactly equal. Call this the *attraction point*.

Find the attraction point. Is there more than one attraction point? Now consider any point in space anywhere. Can any points away from the center line be attraction points? If more attraction points exist, find them, and describe where all the attraction points are located in space.

Comment. You need to look up some basic physical data about the Earth and Moon.

10. Consider the curve $y = x^2$ and a secant line going through points with x-coordinates $x = a$ and $x = 3a$. Where on this curve does the tangent line have the same slope as the secant line?

Comment. A secant line intersects a curve at two points. A tangent line intersects a curve at one point.

11. Use the definition of logarithm,

$$\log_a a^x = x$$

to prove the following:

$$\log_{\frac{1}{a}} b = -\log_a b$$

$$\log_{a^k} b = \frac{1}{k} \log_a b$$

$$log_{\frac{1}{a}} \frac{1}{b} = \log_a b$$

$$\log_a b = \frac{1}{\log_b a}$$

$$\frac{\log_a n}{\log_a m} = \log_m n$$

$$\log_m a = (\log_m n) \log_n a.$$

Comment. All students must know how to prove these. Memorizing them won't help with competition problems. Proving them will.

12. Prove that

$$x^5 - x^4 + 3x^3 - 3x^2 + x - 1 = 0$$

has exactly one positive solution and find it.

13. Why do liquids cool when they evaporate? Try to give an explanation using the idea that matter is made of atoms.

14. Water is an unusual substance for many reasons. One reason is that water expands when it freezes. Why does this happen? Can you explain it using the atomic theory of matter? Can you find other substances that expand when they freeze?

15. Consider four-digit numbers which are multiples of 13. How many are there? What is the sum of all of them?

Hint. Use arithmetic sequences.

16. There are n people in a room. Prove that at least two of them have the same number of acquaintances from among the people in the room.

Comment. This is a fantastical application of Dirichlet's principle. Once you see how it works to solve this problem, it will hit you like mathematical lightning.

17. Consider a sequence of n integers. They don't have to be all different. Prove that there exists a subsequence that is divisible by n.

Hint. Use modular arithmetic and Dirichlet's principle.

18. Consider two situations. (A) an object begins with zero velocity at height h and goes down an inclined plane until it reaches the bottom with velocity v_1. (B) The same object begins at the bottom of the inclined plane and goes up with a velocity v_2 such that when it reaches height h, its velocity is zero. Which velocity is bigger, v_1 or v_2? Consider this problem with and without friction.