Trigonometry and Constructive Geometry

Training problems for M2 2018 term 1

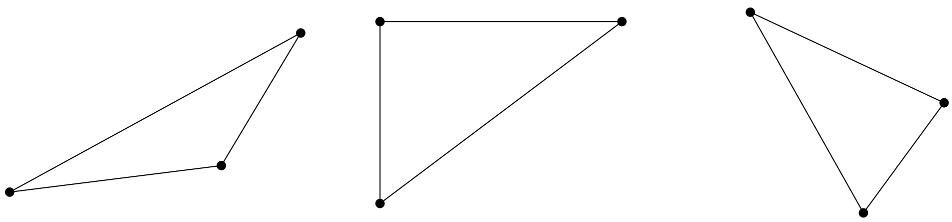
Ted Szylowiec
tedszy@gmail.com

1 Labeling geometrical figures

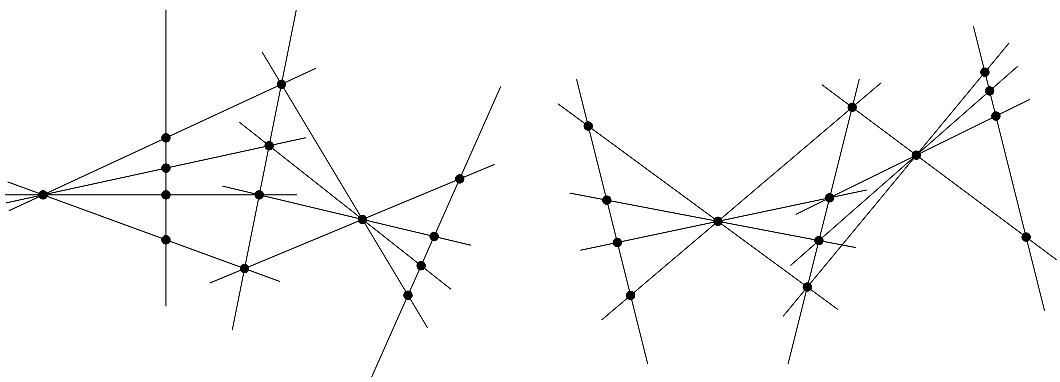
1. Practice writing Greek letters.

$\alpha\beta\gamma\delta\epsilon\theta\lambda\mu\pi\phi\psi$

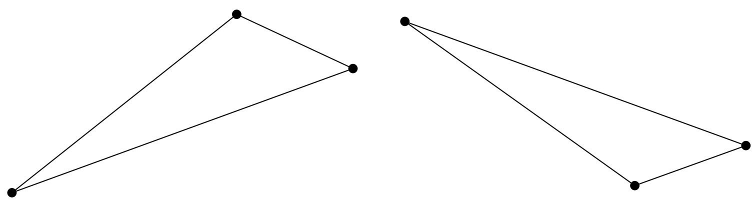
2. Label the sides, angles and vertices of these triangles using the classical method, in counterclockwise order.



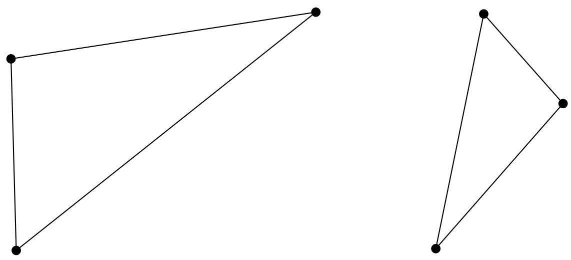
3. Use primes to label the figure in a logical way.



4. These triangles are congruent. Label them using the classical method. Use primes. Write down relationships between angles and sides.

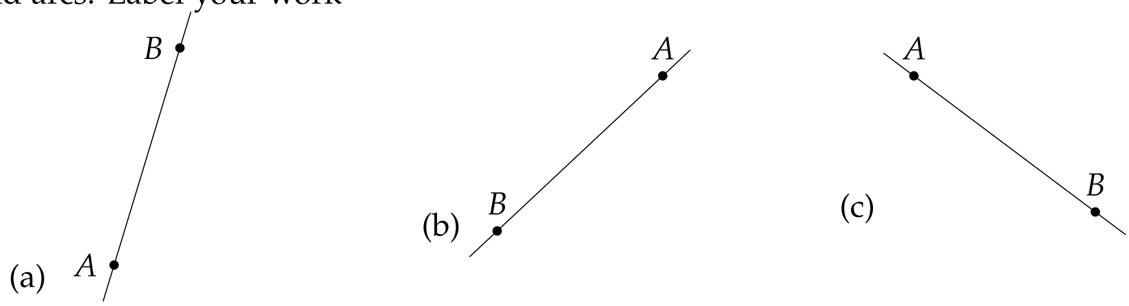


5. These triangles are similar. Label them using the classical method. Write down relationships between angles and between sides. What is the zoom factor? Is it bigger or smaller than 1?

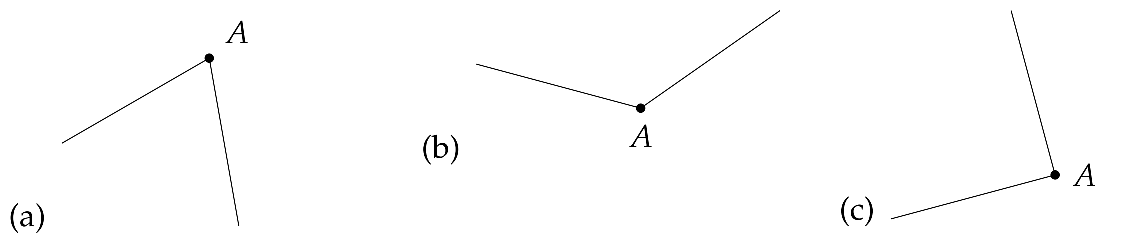


2 Congruence and similarity

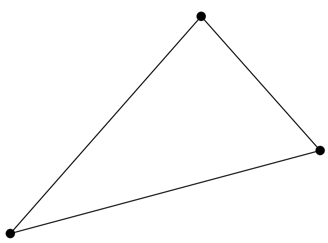
6. Copy these segments using ruler and compass. Don't erase your construction lines and arcs. Label your work



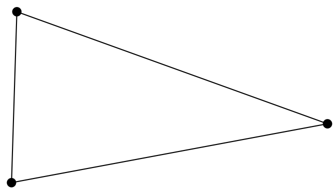
7. Use a ruler and compass to copy the angle at A. Don't erase your arcs or construction lines. Label your work.



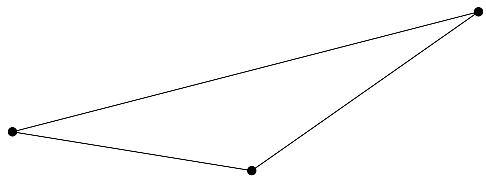
8. Make a congruent copy of this triangle by SSS. Use ruler and compass. Don't erase your construction lines. Label your work



9. Use ruler and compass to make a congruent copy of this triangle by SAS. Label your work. Explain which sides and angle you have copied.



10. Use ruler and compass to make a congruent copy of this triangle by ASA. Label your work. Explain which side and angles you have copied.



11. Use a ruler and compass to construct a counterexample for AAA. Construct two triangles where AAA is true, but not conguent. Use ruler and compass. Label your triangles and write down all the relationships. Is the zoom factor bigger or smaller than 1?

12. Give a counterexample for ASS, SSA. Show that having ASS true leads to two solutions, one congruent, the other not congruent. Use a ruler and compass. Don't erase your construction lines.

13. Prove the parallelogram area formula

$$\text{area} = \text{base} \times \text{height}$$

by doing these steps:

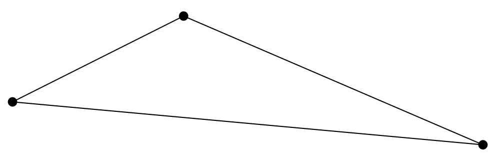
(a) Construct a parallelogram by ruler and compass.
(b) Cut the parallelogram into two triangles.
(c) Copy the two triangles into separate figures using ruler and compass. Label them using classical labelling and primes.
(d) Use SSS to prove that the two triangles are congruent. Explain why each step is true.
(e) Write a conclusion.

14. Do the same parallelogram proof as in problem **13** but using SAS.

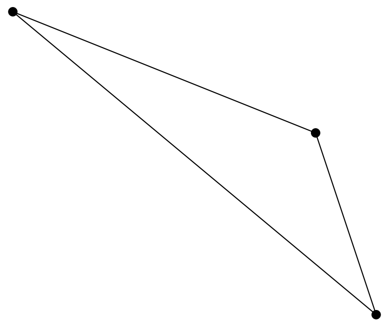
15. Prove the paralellogram area formula using AAS. Follow the steps of problem **13**.

16. Prove the parallelogram area formula using ASA. Follow problem **13**.

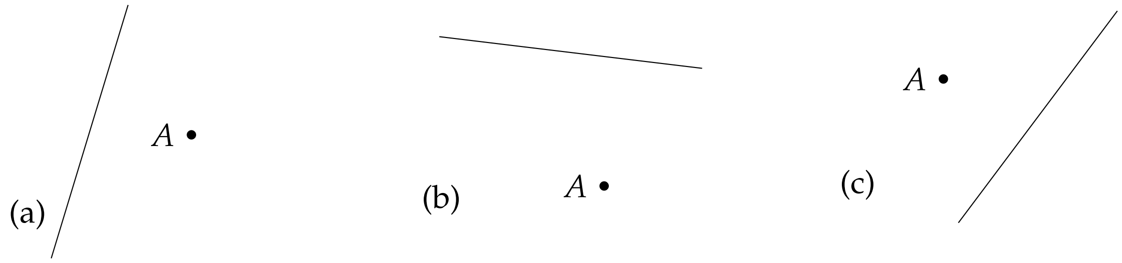
17. Use AAA to make a *smaller* similar copy of this triangle. Do it with ruler and compass. Label your work. Write down the relationships between sides and angles.



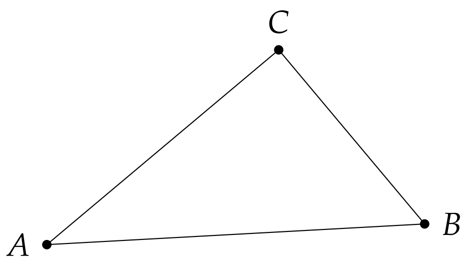
18. Use AAA to make a *larger* similar copy of this triangle. Use ruler and compass. Label your work. Write down relationships between sides and angles.



19. Construct 90° perpendicular lines going through point A. Use ruler and compass. Don't erase construction lines and arcs.



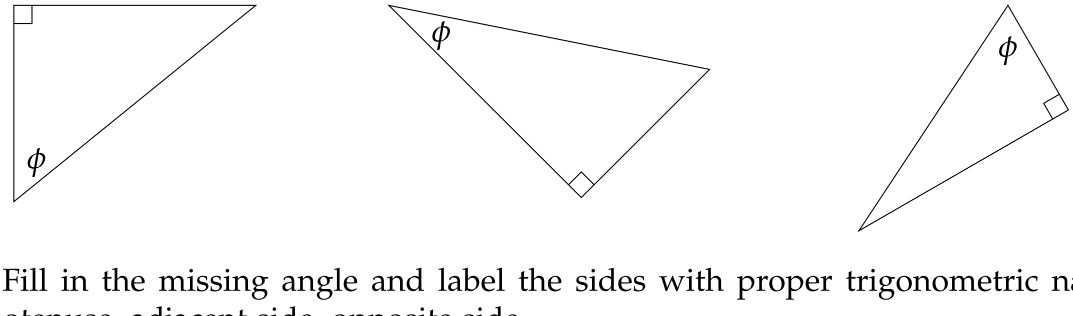
20. Here is a right (90°) triangle. We usually label the vertex with the right angle as C and the longest side as c. Label the triangle and construct the altitude line h at C using ruler and compass.



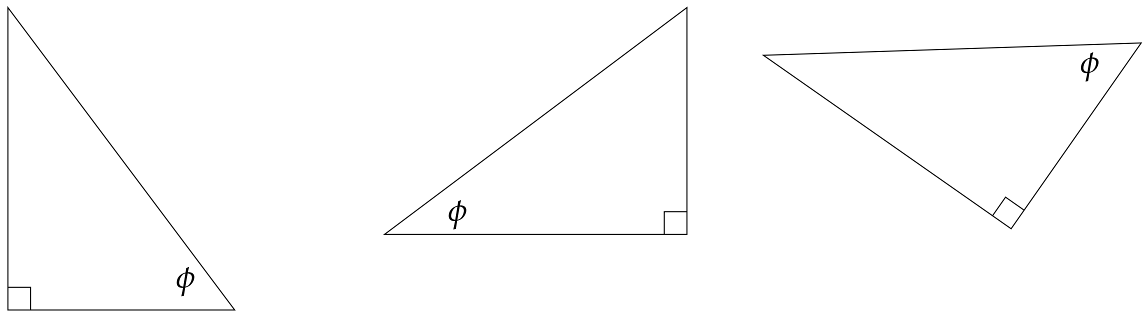
21. Construct a 90° triangle using ruler and compass. Let C be the 90° vertex. Also construct the altitude line at C. Label the vertices, angles and sides of your figure. Don't erase your arcs or construction lines.

3 ϕ–90° triangles

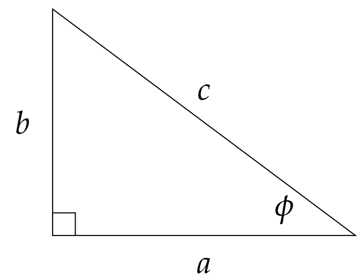
22. Fill in the missing angle.



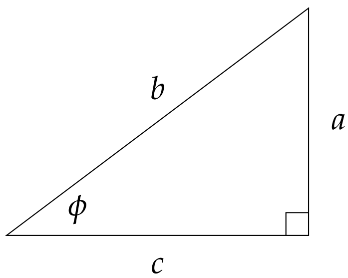
23. Fill in the missing angle and label the sides with proper trigonometric names: hypotenuse, adjacent side, opposite side.



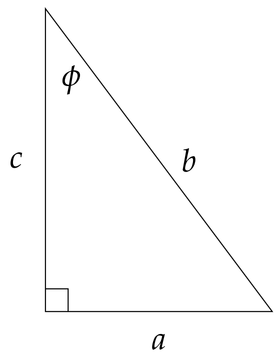
24. Here is a ϕ–90° triangle. What is special about side a? Make a list of things.



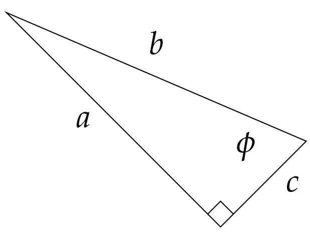
25. Here is a ϕ–90° triangle. What is special about side c? Make a list of things.

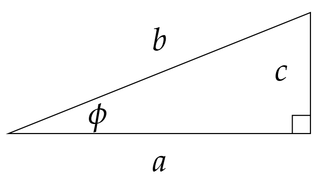


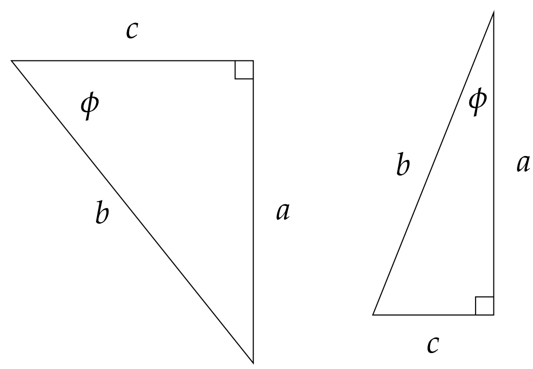
26. Here is a ϕ–90° triangle. What is special about side b? Make a list of things.



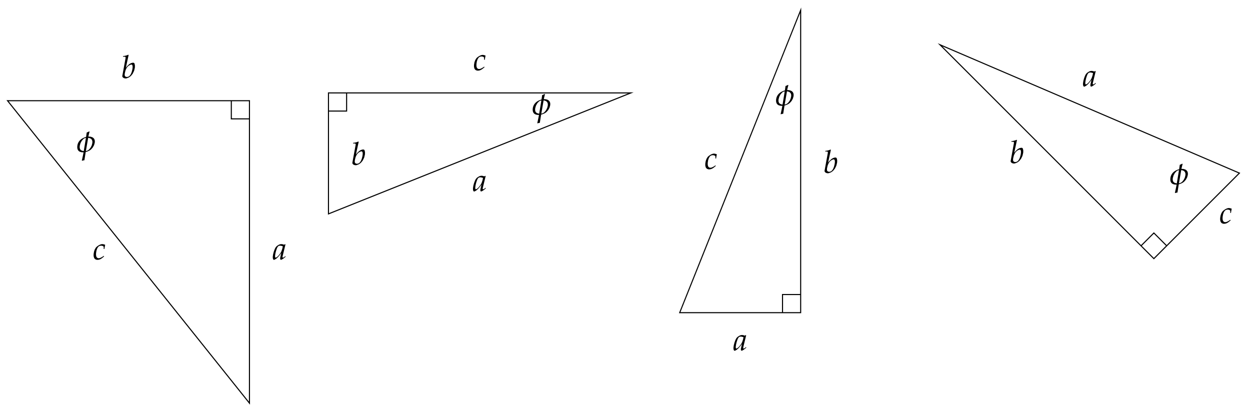
27. Find sin, cos, tan in terms of sides a, b, c.



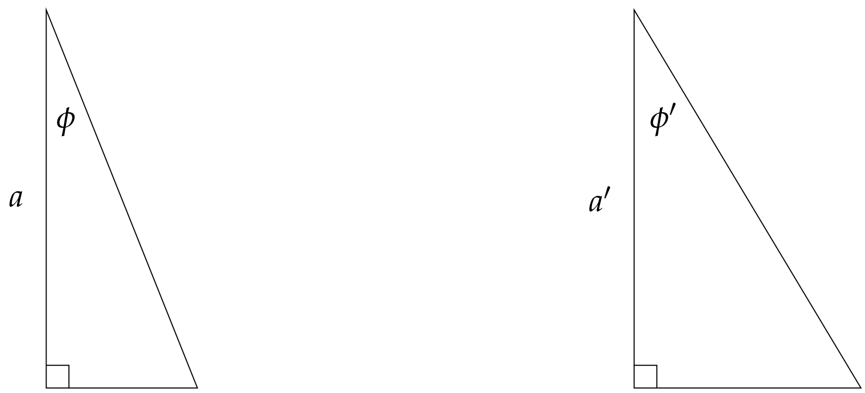




28. Find csc, sec, cot in terms of sides a, b, c.



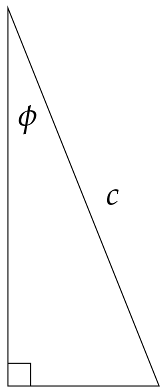
29. Consider the following triangles where $a = a'$ and $\phi < \phi'$.

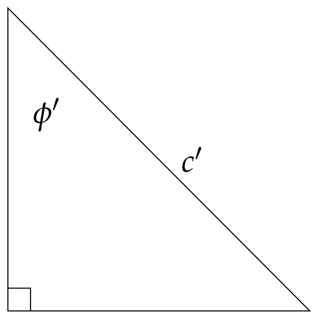


(a) Which is bigger, $\tan\phi$ or $\tan\phi'$?
(b) Which is bigger, $\cot\phi$ or $\cot\phi'$?

Explain why!

30. Consider the following triangles where $c = c'$ and $\phi < \phi'$.

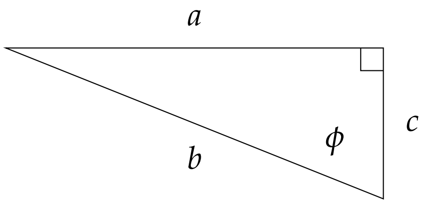


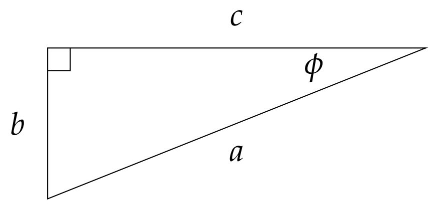


(a) Which is bigger, $\sin\phi$ or $\sin\phi'$?
(b) Which is bigger, $\cos\phi$ or $\cos\phi'$?

Explain why!

31. Label the missing angle, label all sides with proper names and find all trigonometric ratios sin, cos, tan, csc, sec, cot. in terms of sides a, b, c.





32. Why do we have six of these trigonometric ratios, sin, cos, tan, csc, sec and cot? Why are they important? What is so special about them?

33. Explain how we were able to calculate the distance to the star 61 Cygni by using trigonometry.

34. Draw a ϕ–90° triangle, label it, and find a relationship between sin, cos and tan.

35. Draw a ϕ–90° triangle and label it. Use Pythagoras's law to find a relationship between sin, cos and 1.

36. Draw a ϕ–90° triangle and label it. Use Pythagoras's law to find a relationship between tan, sec and 1.

37. Draw a ϕ–90° triangle and label it. Use Pythagoras's law to find a relationship between cot, csc and 1.

4 Special angles

38. Change these angles from radians into degrees. Figure it out by drawing little circles.

(a) $\pi/2$. (c) 7π. (e) $3\pi/4$. (g) $2\pi/3$.
(b) $\pi/6$. (d) $5\pi/8$. (f) $5\pi/12$. (h) $3\pi/2$.

39. Change these angles from degrees into radians. Figure it out by drawing little circles.

(a) $75°$. (c) $285°$. (e) $225°$. (g) $195°$.
(b) $300°$. (d) $120°$. (f) $720°$. (h) $22.5°$.

40. Sketch a $\pi/6$–$\pi/3$–$\pi/2$ triangle. Make the hypotenuse 1. Label all the angles and the lengths of the sides.

41. Sketch a $45°$–$45°$–$90°$ triangle. Make the hypotenuse 1. Label all the angles and the lengths of the sides.

42. Check that $(\sin\phi)^2 + (\cos\phi)^2 = 1$ for $\phi = 30°$.

43. Check that $(\sec\phi)^2 - (\tan\phi)^2 = 1$ for $\phi = \pi/4$.

44. Check that $(\csc\phi)^2 - (\sec\phi)^2 = 1$ for $\phi = 60°$.

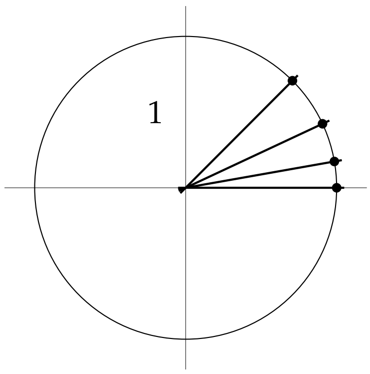
5 The unit circle diagram

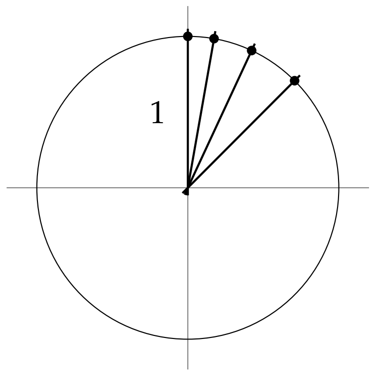
Always remember: the unit circle has radius 1, even if that is not shown explicitly in the diagram.

45. What does *unit* mean in the term *unit circle*?

46. Construct (with ruler and compass): axes, unit cicle, angle ϕ.

47. Sketch the sine and cosine lines and label them.



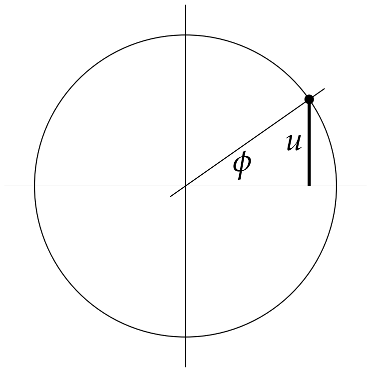


48. Construct with ruler and compass: axes, the unit circle, angle ϕ, cosine line and sine line.

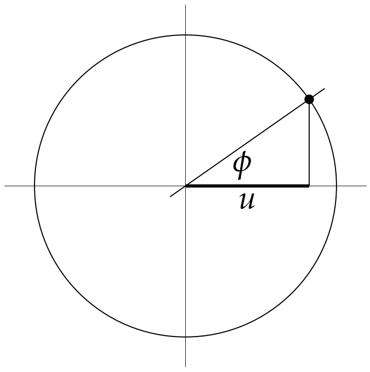
49. What is the behavior of sin and cos as ϕ goes to 0 and $90°$?

angle	sin	cos
$\phi \to 0$		
$\phi \to 90°$		

50. Prove that the length u is the same as $\sin\phi$.



51. Prove that length u is $\cos\phi$.

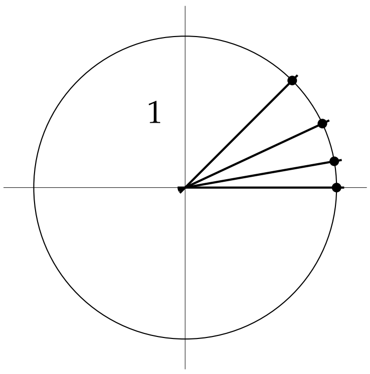


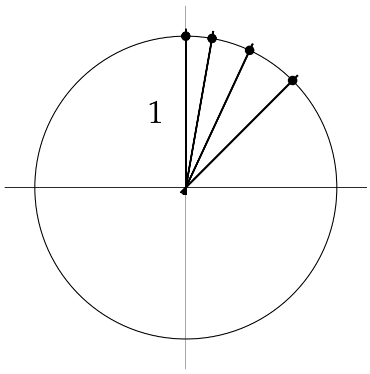
52. Interpret the meaning of

$$(\cos\phi)^2 + (\sin\phi)^2 = 1$$

using the unit circle diagram.

53. Sketch the tan and sec lines and label them.



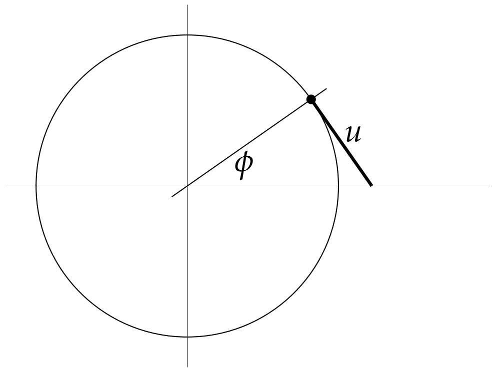


54. Construct using ruler and compass: axes, unit circle, angle, tangent line and secant line.

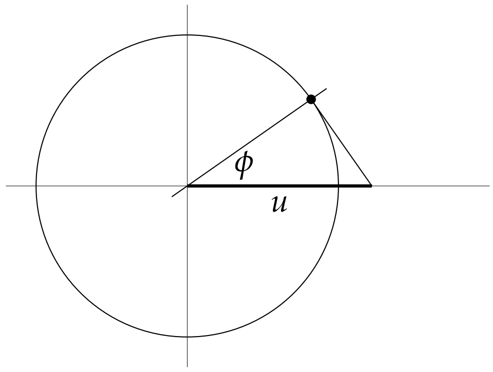
55. What is the behavior of tan and sec as ϕ goes to 0 and $90°$?

angle	tan	sec
$\phi \to 0$		
$\phi \to 90°$		

56. Prove that length u is $\tan\phi$. Use similar triangles.



57. Prove that length u is $\sec\phi$. Use similar triangles.

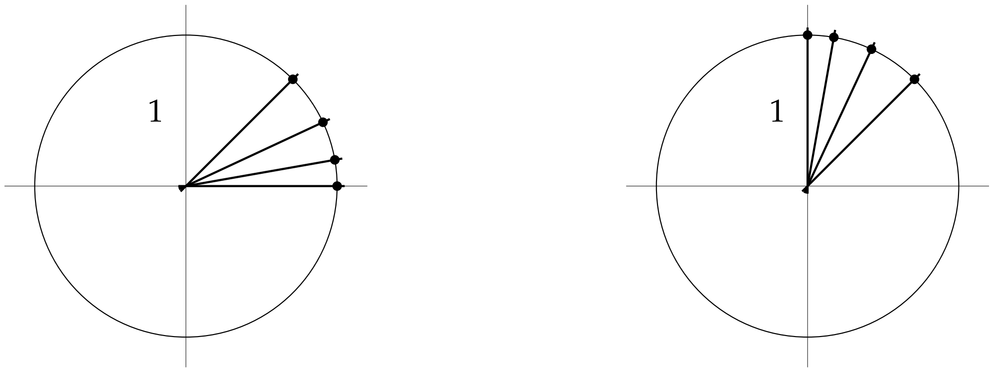


58. Interpret the meaning of

$$(\sec\phi)^2 - (\tan\phi)^2 = 1$$

using the unit circle diagram.

59. Sketch the cosecant and cotangent lines and label them.

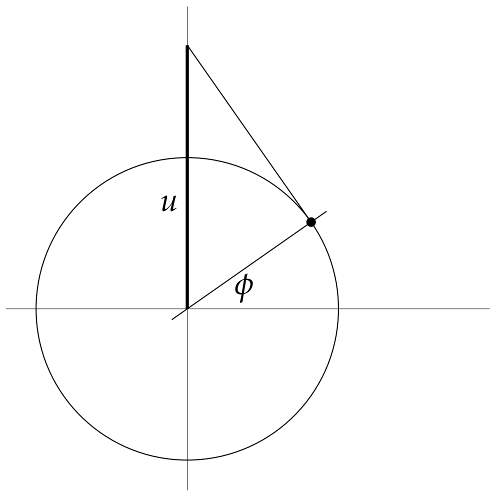


60. Use a ruler and compass to construct axes, the unit circle, angle, cosecant line and cotangent line.

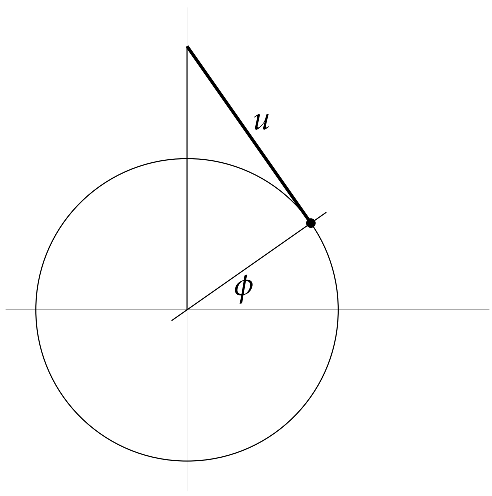
61. What is the behavior of csc and cot as ϕ goes to 0 and 90°?

angle	csc	cot
$\phi \to 0$		
$\phi \to 90^\circ$		

62. Prove that u is $\csc\phi$. Use similar triangles.



63. Prove that u is $\cot\phi$. Use similar triangles.



64. Interpret the meaning of

$$(\csc\phi)^2 - (\cot\phi)^2 = 1$$

using the unit circle diagram.

65. Organize the six trigonometric functions according to their relationship to the unit circle.

Inside the circle:	
Outside the circle:	
Partly in and partly out:	

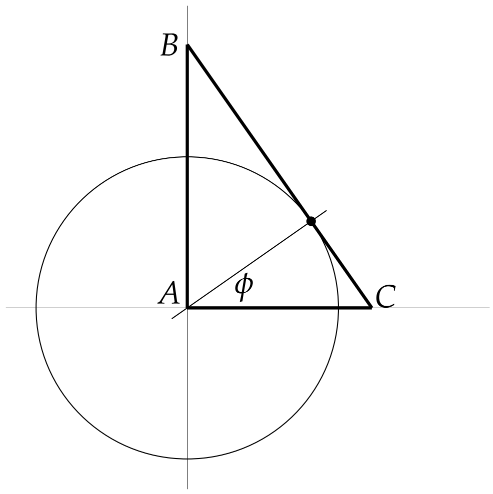
66. Fill in this table of trig function behaviors. It's easy if you keep the unit circle diagram in mind, and use your imagination.

$\sin \to 0$	$\cos \to ?$
$\cos \to ?$	$\sin \to 1$
$\sec \to 1$	$\csc \to ?$
$\csc \to ?$	$\sec \to \infty$
$\tan \to 0$	$\cot \to ?$
$\cot \to ?$	$\tan \to \infty$

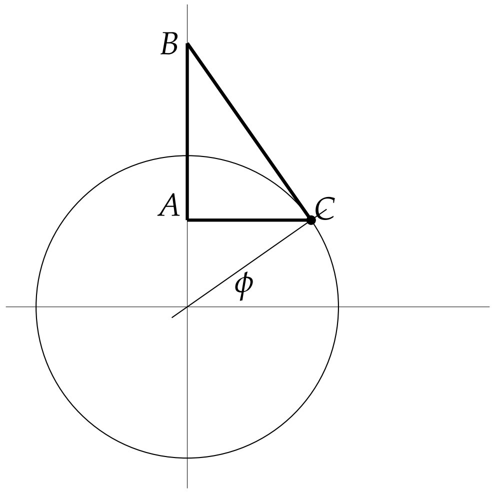
6 Trigonometric identities

67. What do we mean by a *trig identity*? Give some examples.

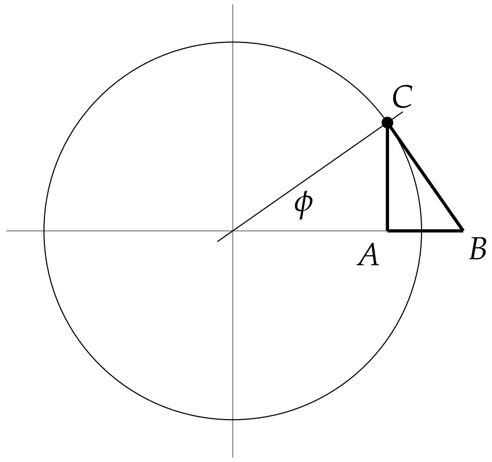
68. Apply Pythagoras's law to triangle ABC. Do you get anything new? Any new identities or relationships between trig functions?



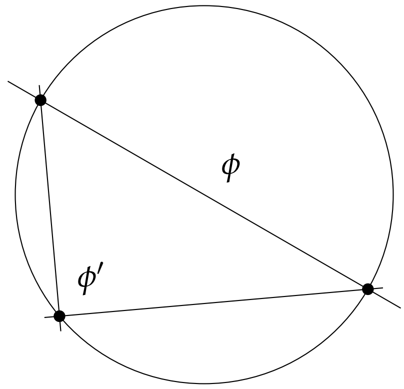
69. Apply Pythagoras's law to triangle ABC. Do you get anything new?



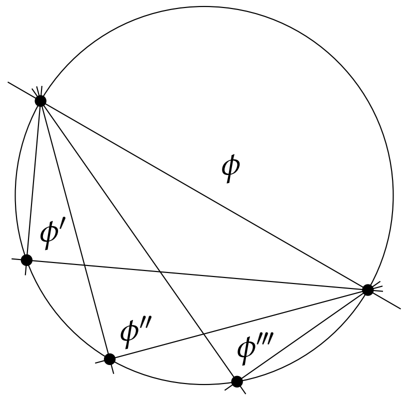
70. Apply Pythagoras's law to triangle ABC. Do you get anything new?



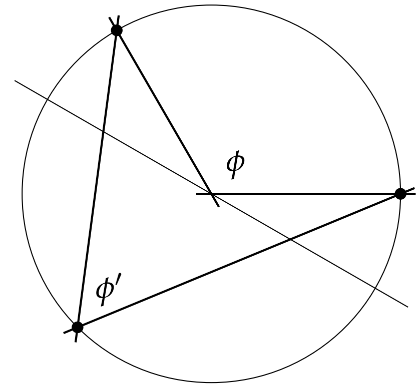
71. Given that the line through the circle is a center line. What are the angles ϕ and ϕ'?



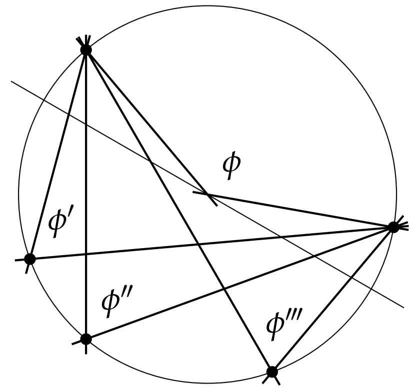
72. What are the angles ϕ, ϕ', ϕ'', ϕ'''? The line through the circle is a center line.



73. What is the relationship between angles ϕ and ϕ'?



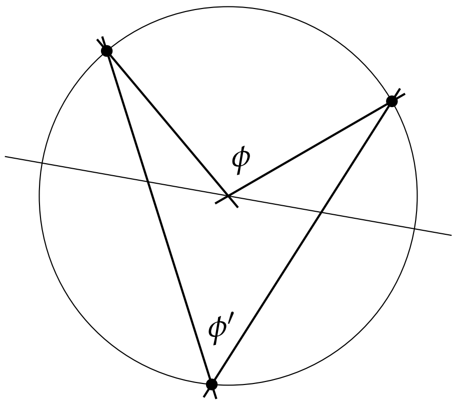
74. What are the relationships between angles ϕ, ϕ', ϕ'', ϕ'''?



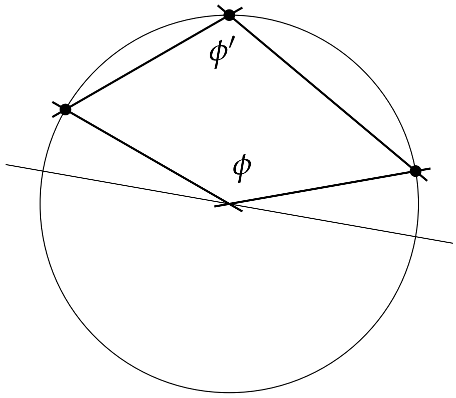
75. Use SSS congruence to prove that if a triangle has two equal sides, then it has two equal angles.

76. Prove the very beautiful circle-angle theorem:

$$\phi' = \frac{\phi}{2}.$$



77. And now what happens when the angle ϕ' is on the other side? Prove a relationship between ϕ and ϕ' using the same ideas as you used in problem **76** and in problem **75**.



78.