M2 Training Problems

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1 Functions, identity, inverses and plots

1. Let $f(x) = 2x + 1$. Find...
  (a) Find $f(f(x))$.
  (b) Find $f(f(f(x)))$.
  (c) Find $f(f(f(f(x))))$.

2. Let $f(x) = 3x^2 + 1$ and $g(x) = 2x - 3$.
  (a) Find $f(g(x))$.
  (b) Find $g(f(x))$.
Are they the same?

3. Let $f(x) = ax + b$.
  (a) Find $f(f(x))$.
  (b) Find $f(f(f(x)))$.

4. Let $f(x) = ax + b$ and $g(x) = cx + d$.
  (a) Find $f(g(x))$.
  (b) Find $g(f(x))$.
Are they the same?

5. Sketch $y = x$ and $y = -x$. Put them on the same axes. Label everything.

6. Sketch $y = 2x$ and $y = -2x$. Put them on the same axes.

7. Sketch these lines on the same axes.

$$y = \frac{x}{2}, \quad y = -\frac{x}{2}.$$

8. Make an exact plot of $y = 3x + 2$ by finding the x-intercept and y-intercept.

9. Make an exact plot of

$$y = -\frac{x}{2} - 1.$$

10. If $f(x)$ and $g(x)$ are linear, show that
  (a) $f(g(x))$ is linear.
  (b) $g(f(x))$ is linear.

11. Let $f(x) = 3x + 2$. Find $f^{-1}(x)$. Do it two ways:

(a) By $f(f^{-1}(x)) = I(x)$.
(b) And by $f^{-1}(f(x)) = I(x)$.

12. Let $f(x) = ax + b$. Find $f^{-1}(x)$. Do it two ways:
(a) By $f(f^{-1}(x)) = I(x)$.
(b) And by $f^{-1}(f(x)) = I(x)$.

13. Let $f(x) = 2x + 1$. Find $f^{-1}(x)$. Make exact plots of f and f^{-1}. Also draw I.

14. Let $f(x) = -2x + 3$. Find $f^{-1}(x)$. Make exact plots of f and f^{-1}. Also draw I.

15. Consider the function
$$f(x) = -\frac{x}{2} + 3.$$
Find $f^{-1}(x)$. Make exact plots of f and f^{-1}. Also draw I.

16. Sketch the curve $y = x^2$. Use the unit square idea.

17. Let $f(x) = x^2$. Sketch f, I and f^{-1} on the same axes.

18. Let $f(x) = x^2 + 1$. Sketch f, I and f^{-1} on the same axes.

19. Are there functions that are inverses of themselves? Does there exist any functions with the property $f(x) = f^{-1}(x)$? In other words, f is its own inverse.
(a) Find one such self-inverse function f.
(b) Try to find more, as many as you can.

2 Introducing logarithms

20. Draw the fastest-growing function f that you can imagine. Draw $I(x)$ and use it to find the inverse f^{-1}.

21. Draw the slowest-growing function f that you can imagine. Draw the identity $I(x)$ and use it to find the inverse f^{-1}.

22. Given f, tell me about the inverse f^{-1}. Does it grow fast, slowly, very fast etc.?
(a) f is a fast-growing function.
(b) f does not grow at all.
(c) f is a slow-growing function.
(d) f is a very slow-growing function.
(e) f is a very fast-growing function.

23. Fill in this table about the behavior of $f(x) = 2^x$ for different values of x.

x	$f(x)$
$x = 0$	$f = 1$
$x > 0$	
$x < 0$	
$x > 1$	
$x \to \infty$	
$x \to -\infty$	

24. Plot 2^x, 3^x and 5^x all on the same axes.

25. Consider the function $f(n) = \left(1 + \frac{1}{n}\right)^n$. Use a calculator. Fill in this table

x	$f(x)$
1	2
2	
5	
10	
100	
1000	

Notice how $f(n)$ keeps increasing as n gets bigger. But also notice how $f(n)$ does not increase to infinity, but approaces the magic number *e from below*.

26. Now consider the slightly different function $g(n) = \left(1 + \frac{1}{n}\right)^{n+1}$. Use a calculator. Fill in this table

x	$f(x)$
1	2
2	
5	
10	
100	
1000	

Notice how $g(n)$ keeps decreasing as n gets bigger. But also notice how $g(n)$ does not decrease to minus infinity, but approaces the magic number *e from above*.

27. Plot 2^x, e^x and 10^x all on the same axes.

28. Plot $f(x) = 2^x$ and the identity line $I(x)$. Use the identity line to draw the inverse $f^{-1}(x) = \log_2 x$.

29. Plot $f(x) = 3^x$, $I(x)$ and $f^{-1}(x) = \log_3 x$ all on the same axes.

30. Fill in this table about the behavior of $g(x) = \log_2 x$ for different values of x.

x	$g(x)$
$x = 1$	$g = 0$
$x > 1$	
$x < 1$	
$x = 2$	
$x \to \infty$	
$x \to 0$	

31. Does 2^x ever touch the x-axis? Does $\log_2 x$ ever touch the y-axis?

32. Fill in the table.

x	3^x
1	
2	
3	
4	
5	

x	$\log_3 x$
1	
9	
27	
243	
59049	

33. Fill in the table.

x	10^x
1	
2	
3	
4	
5	

x	$\log_{10} x$
1	
10	
1000	
$100,000$	
$10,000,000$	

34. Plot $\log_2 x$, $\log_3 x$ and $\log_5 x$ all on the same axes. Label all the important points.

35. Plot $\log_2 x$, $\log_e x$ and $\log_{10} x$ all on the same axes. Label all the important points.

36. The formulas relating f, f^{-1} and I establish the two most important properties of logarithms and exponentials. Use $f(x) = a^x$ and $f^{-1}(x) = \log_a x$ and tell me what these formulas imply:

(a) $f(f^{-1}(x)) = I(x)$.
(b) $f^{-1}(f(x)) = I(x)$.

3 Properties of $\log_a$

37. Figure out $a^{\log_a a^x}$.

38. Figure out $\log_a a^{\log_a a^x}$.

39. Begin with the well-known property of exponential functions $(a^x)^p = a^{xp}$ and prove the following property of logarithms:

$$\log_a u^p = p \log_a u.$$

Notice how log changes powers to multiplications.

40. Begin with something we all know: $a^x a^y = a^{x+y}$ and prove the following property of logarithms:

$$\log_a uv = \log_a u + \log_a v.$$

Notice how log changes multiplication into addition.

41. Using an idea similar to the one in problem **40**, prove that

$$\log_a uvw = \log_a u + \log_a v + \log_a w.$$

42. Use the results of problems **39** and **40** to prove this:

$$\log_a \frac{u}{v} = \log_a u - \log_a v.$$

Notice how log changes division into subtraction.

43. Use **39**, **40** and **42** to figure these out.

(a) $\log_2 8 \times 32 \times 64$.
(b) $\log_5 25 \times 125 \times 625$.
(c) $\log_2 2^7 8^5 16^3$.
(d) $\log_3 \sqrt{27}\sqrt[3]{81}$.
(e) $\log_5 \dfrac{\sqrt{125}}{\sqrt[3]{625}}$.
(f) $\log_e \dfrac{\sqrt{e}}{e^3}\sqrt[3]{e}$.

44. Use plots to explain why $0 < x < 1$ when $\log x$ is negative.

45. Use plots to explain why $\log x$ is positive when $x > 1$.

46. Let $x = a/b$. Use the formula in **42** to show that $\log x$ is negative when $0 < x < 1$.

47. Let $x = a/b$. Use the formula in **42** to show that $\log x$ is positive when $x > 1$.

48. Positive or negative? Use log, algebra and plots to explain why. Assume the logarithm base is $a > 1$ as usual.

(a) $\log_a \frac{2}{3}$

(b) $\log_a \frac{5}{2}$

(c) $\log_a 0.8$

(d) $\log_a 1.8$

(e) $\log_a \frac{e}{\pi}$

(f) $\log_a \frac{\pi}{e}$

49. Positive or negative? Explain why.

(a) $\log \frac{\pi}{\sqrt{10}}$

(b) $\log \frac{\sqrt{10}}{\pi}$

(c) $\log \frac{3e}{2\pi}$

(d) $\log \frac{2\pi}{3e}$

50. Bigger or smaller than 1? Explain using logarithms, algebra and plots.

(a) $\left(\frac{3}{2}\right)^{2/3}$

(b) $\left(\frac{2}{3}\right)^{-3/2}$

(c) $\left(\frac{\pi}{5}\right)^{\sqrt{5/2}}$

(d) $\left(\frac{1}{\sqrt{3}}\right)^{-1/\sqrt{2}}$

(e) $(\sqrt{2})^{-\sqrt{2}}$

(f) $\left(\frac{1}{\pi}\right)^{1/e}$

(g) $(\sqrt{e})^{-\sqrt{\pi}}$

51. Given the inequality, determine which is bigger: x or y. Prove it using logarithms.

(a) $(0.5)^x > (0.5)^y$.

(b) $(1.5)^x < (1.5)^y$.

(c) $\left(\frac{3}{2}\right)^x > \left(\frac{3}{2}\right)^y$.

52. Which is bigger: x or y?

(a) $\left(\frac{2}{3}\right)^x < \left(\frac{2}{3}\right)^y$.

(b) $\left(\frac{e}{\pi}\right)^x > \left(\frac{e}{\pi}\right)^y$.

(c) $\left(\frac{\pi}{e}\right)^x < \left(\frac{\pi}{e}\right)^y$.

53. Which is bigger? Explain using logarithms, plots, algebra etc.

(a) 2^{70} or 7^{20}?

(b) 5^{30} or 3^{50}?

(c) 2^{50} or 5^{20}?

54. Simplify.

(a) $\dfrac{\log_2 81}{\log_2 27}$

(b) $81^{\log_3 2}$

(c) $64^{\log_4 3}$

(d) $e^{\log_e(\log_e e^2)}$

(e) $\left(\frac{1}{100}\right)^{\log_{10} 2}$

55. Prove that

$$\log_a b = \frac{1}{\log_b a}.$$

56. Simplify.

(a) $a^{1/\log_b a^2}$

(b) $a^{1/\log_{b^2} a}$

(c) $a^{1/\log_{b^6} a^3}$

(d) $3^{1/\log_5 3}$

(e) $3^{1/\log_5 3^2}$

(f) $2^{1/\log_{27} 8}$

57. Simplify and explain what values of a and b are possible.

(a) $\log(ab) - \log|a|$.

(b) $\log(ab) - \log|b|$.

(c) $\log(ab) - \log|a| - \log|b|$.

58. Prove that

$$\log_{a^2} x = \frac{1}{2}\log_a x.$$

59. Prove that

$$\log_{a^p} x = \frac{1}{p}\log_a x.$$

60. Simplify. $\log_a b^2 + \log_{a^2} b^4 + \cdots + \log_{a^n} b^{2n}$.

61. Simplify. $(\log_a b)(\log_b c)(\log_c a)$.

62. Simplify. $(\log_a b)(\log_b c)(\log_c d)(\log_d a)$.

63. Simplify. $(\log_a b)(\log_{b^2} c^2)(\log_{c^3} a^3)$.

64. Simplify. $(\log_a b)(\log_{b^2} c^2)(\log_{c^3} d^3)(\log_{d^4} a^4)$.

65. Simplify. $(\log_a b^4)(\log_{b^2} c^3)(\log_{c^3} d^2)(\log_{d^4} a)$.

66. Simplify. $\dfrac{\log_8 5}{\log_4 25}$.

67. Simplify. $(\log_8 5)(\log_{125} 4)$.

68. Simplify. $\log_{\sqrt{a}} \sqrt{x}$.

69. Simplify. $\left(\log_{\sqrt{a}} b\right)\left(\log_{\sqrt{b}} a\right)$.

70. Simplify. $(\log_7 2)(\log_3 5)(\log_5 7)(\log_2 3)$.

71. Simplify. $(\log_5 3)(\log_4 5)(\log_{125} 8)$.

72. Simplify $(\log_5 2)(\log_{27} 125)(\log_2 3)$.

73. Use $\sqrt{\log_a b}\sqrt{\log_a b} = \log_a b$ and prove that

$$a^{\sqrt{\log_a b}} = b^{\sqrt{\log_b a}}.$$

For what values of a and b is this true?

74. Prove the famous change-of-base formula:

$$\log_a x = (\log_a b)\log_b x.$$

4 $\log_a$ with $0 < a < 1$. What happens?

75. Show that $\log_1 x$ is undefined. (a) Use algebra and log properties. (b) Use plots. Plot $y = 1^x$, $I(x)$ and the construct the inverse function $\log_1 x$. Then explain why it is undefined.

76. Show that $\log_0 x$ is undefined. (a) Use algebra and basic log properties. (b) Use plots.

77. Use algebra fundamental properties of logarithm to prove that $\log_a x$ is undefined when $a < 0$.

78. Let $a > 1$. Consider the sequence a, a^2, a^3, $a^4 \ldots$ Are these numbers increasing or decreasing?

79. Let $0 < a < 1$. Consider the sequence $a, a^2, a^3, a^4 \ldots$ Are these numbers increasing or decreasing?

80. Let $a > 1$. Consider the sequence $a, a^{-2}, a^{-3}, a^{-4} \ldots$ Are these numbers increasing or decreasing?

81. Let $0 < a < 1$. Consider the sequence $a, a^{-2}, a^{-3}, a^{-4} \ldots$ Are these numbers increasing or decreasing?

82. Plot $y = a^x$ for $a > 1$.

83. Plot $y = a^x$ for $0 < a < 1$.

84. Plot on the same axes $y = a^x$, $I(x)$ and $\log_a x$ for $a > 1$. Use reflection across $I(x)$ to get $\log_a x$.

85. Plot on the same axes $y = a^x$, $I(x)$ and $\log_a x$ for $0 < a < 1$. Use reflection across $I(x)$ to get $\log_a x$.