SME M2 Training Problems

Ted Szylowiec

tedszy@gmail.com

Some problems may be harder than usual (or more advanced.) They are marked with a dot •. Problems that are possibly even harder (and even more advanced) are marked with two dots ••. All problems are worth doing! Try to do all of them. Most don't take very long.

Always square

1. Translate this into an algebraic expression: *the sum of the cubes of six consecutive integers.*

2. Even integers have the form $2n$. Translate this into an algebraic expression: *the sum of the squares of five consecutive even integers.*

3. Odd integers have the form $2n+1$. Translate this into an algebraic expression: *the sum of the fourth powers of three consecutive odd integers.*

4. Show that $n^4+10n^3+27n^2+10n+1$ is a perfect square for all n. Do it by writing it in the form $(an^2+bn+c)^2$ and finding the coefficients a, b, c.

5. Show that $n^4-14n^3+51n^2-14n+1$ is a perfect square.

6. Show that $n^4+14n^3+51n^2+14n+1$ is a perfect square.

7. • Show that $n^6+2n^5+5n^4+6n^3+6n^2+4n+1$ is a perfect square.

8. • Show that $n^6+10n^5+27n^4+12n^3+11n^2+2n+1$ is a perfect square.

9. • Show that $n^6+4n^5+10n^4+14n^3+13n^2+6n+1$ is a perfect square.

10. Consider positive integers in steps of five: n, $n+5$, etc. Let ϕ be the product of four such consecutive numbers. Prove that $\phi+625$ is always a perfect square.

11. Consider positive integers in steps of 10: n, $n+10$, etc. Let ϕ be the product of four such consecutive numbers. Prove that $\phi+10000$ is always a perfect square.

12. • Even numbers have the form $2n$. Let ϕ be the product of four consecutive even numbers. Show that $\phi+16$ is always a perfect square.

13. • Odd numbers have the form $2n+1$. Let ϕ be the product of four consecutive odd numbers. Prove that $\phi+16$ is always a perfect square.

14. • Examine the patterns in problems **10**, **11**, **12** and **13**. Can you combine all these results into one mathematical statement?

15. • Prove the answer you gave to problem **14**.

Never square

16. Draw $17 \mod 4$ using a dot drawing.

17. If $x \equiv 2 \mod 5$ and $y \equiv 3 \mod 5$ then show that 5 divides $x + y$ by using dot drawings.

18. Find the last digit of 3^{20}. Use modular arithmetic. Don't use a calculator.

19. What is the last digit of 7^{101}? Use modular arithmetic.

20. Use negative residues to find $36^5 - 6^{50} + 48^{100} \mod 7$. Hint: use $48 = 6 \times 8$.

21. Use modular arithmetic to show that 33 divides $2^{50} - 1$. Use negative residues.

22. Use negative residues to prove that 10 divides $3^{20} - 1$.

23. Prove that 15 divides $2^{100} - 1$.

24. Prove that 6 divides the product of three consecutive integers.

25. Prove that 10 divides the product of five consecutive integers.

26. Prove that 35 divides the product of seven consecutive integers.

27. Prove that 7 always divides $n^7 - n$ for any integer n.

28. Prove that it is not true that 9 always divides $n^9 - n$. Use modular arithmetic to find an n such that $9 \nmid n^9 - n$.

29. Use modular arithmetic to prove that $11 \mid n^{11} - n$ for all integers n.

30. Show that 10 does not always divide $n^{10} - n$ and give an example of a number n such that $10 \nmid n^{10} - n$.

31. Is it true that 12 always divides $n^{12} - n$? Use modular arithmetic to find a counterexample.

32. Show that 7 does not divide $7n^3 + 3n^2 + 3n + 5$. Use modular arithmetic.

33. Prove that for all values of n, $121n^3 + 77n^2 + 66n + 11$ is never square.

34. • Prove that the sum of four consecutive fourth powers can never be square.

35. • Prove that the sum of the squares of five consecutive odd numbers is never square.

36. •• Prove that the sum of the fourth powers of four consecutive odd numbers is never square.

37. • • • Open question. Consider every kth integer. Let ϕ be the sum of the squares of five such consecutive integers. Investigate different values of k. Can you prove that it is never square for all k?

38. • • • Open question. Consider again every kth integer. Let ϕ be the sum of the fourth powers of four such consecutive integers. Is ϕ ever square? Can you prove that it can or cannot be square?

Sums of integers

39. Use the reverse method to sum all the integers from 1 to n^2.

40. Sum the first n odd integers beginning with 1. Use Euler's reverse method.

41. Sum the first n even integers beginning with 2. Use Euler's reverse method.

42. Use dot diagrams mod 2 to show how the odd numbers fit together to form a square having n^2 dots. Use this to prove that the sum of the first n odd numbers beginning with 1 is n^2.

43. Use dot diagrams mod 2 to show how the even numbers fit together to form a square without a diagonal, having $n^2 - n$ dots. Use this to prove that the sum of the first $n - 1$ even numbers beginning with 2 is $n^2 - n$.

44. Sum n consecutive integers congruent to 2 mod 3. Begin with 2.

45. Starting with 1, sum n consecutive integers congruent to 1 mod 5.

46. Starting with 4, sum n consecutive integers congruent to 4 mod 5.

47. • Starting with b, sum n consecutive integers congruent to b mod 7.

48. • Starting with b, sum n consecutive integers congruent to b mod a.

49. • Starting with $a + b$, sum n consecutive integers congruent to b mod a.

50. •• In problem **42** you determined the sum of consecutive integers congruent to 1 mod 2 by creating lego-like dot drawing and fitting them together. Can you do this for the sum of n consecutive integers (beginning with 1) congruent to 1 mod 3? You will have to design new kinds of dot drawing legos and fit them together in some way. This is an open question, I myself don't have an answer yet. Some students in other sections claim to have done it.

Sum of Powers

51. Construct Pascal's Triangle from row $n = 0$ to row $n = 10$.

52. Use the $n = 7$ row of Pascal's triangle to expand $(x + y)^7$.

53. Use Pascal's triangle to expand $(1 + y)^5$.

54. Expand by Pascal's triangle: $(x + 1)^8$.

55. Expand using Pascal's triangle: $(3 + 2)^6$.

56. Expand by Pascal's triangle: $(0 + 1)^3$.

57. Expand by Pascal's triangle: $(1 + 1)^4$.

58. Expand by Pascal's triangle: $(5 + 1)^4$.

59. Use the $n = 2$ row of Pascal's triangle to find $S = 1 + 2 + 3 + \cdots + n$.

60. Use the result from **59** and the $n = 3$ row of Pascal's Triangle to find

$$S = 1^2 + 2^2 + 3^2 + \cdots + n^2.$$

61. Label the circled elements in Pascal's triangle with binomial symbols.

$$\begin{array}{c}
\textcircled{1} \\
1 \quad 1 \\
1 \quad 2 \quad \textcircled{1} \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad \textcircled{4} \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad \textcircled{10} \quad 5 \quad \textcircled{1} \\
\textcircled{1} \quad 6 \quad 15 \quad \textcircled{20} \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad \textcircled{21} \quad 35 \quad 35 \quad 21 \quad \textcircled{7} \quad 1 \\
1 \quad 8 \quad 28 \quad \textcircled{56} \quad 70 \quad 56 \quad \textcircled{28} \quad 8 \quad 1
\end{array}$$

62. Draw Pascal's triangle and verify that this is true.

$$\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5} = 2^5.$$

63. Verify that this is true.

$$\binom{8}{0} + \binom{8}{1} + \binom{8}{2} + \binom{8}{3} + \binom{8}{4} + \binom{8}{5} + \binom{8}{6} + \binom{8}{7} + \binom{8}{8} = 2^8$$

64. Draw Pascal's triangle and verify that this is true.

$$\binom{3}{0}^2 + \binom{3}{1}^2 + \binom{3}{2}^2 + \binom{3}{3}^2 = \binom{6}{3}.$$

65. Verify that this is true.

$$\binom{4}{0}^2 + \binom{4}{1}^2 + \binom{4}{2}^2 + \binom{4}{3}^2 + \binom{4}{4}^2 = \binom{8}{4}.$$

66. Verify that this is true.

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} + \binom{6}{2} + \binom{7}{2} = \binom{8}{3}.$$

67. Verify this.

$$\binom{3}{0} + \binom{4}{1} + \binom{5}{2} + \binom{6}{3} + \binom{7}{4} = \binom{8}{4}.$$

68. Use row 4 of Pascal's triangle to find S_3. Show that

$$S_3 = (S_1)^2$$

by factoring the expression for S_3.

69. Find a formula for building S_4 in terms of S_0, S_1, S_2, S_3 and binomial symbols. Do a complete proof using the algebraic method that we learned in class. Use this building formula to get an answer for S_4.

70. Guess a fromula for building S_8 in terms of $S_0, \ldots S_7$ and binomial symbols. You don't have to prove it by the algebraic method. Just guess based on patterns.

71. • Guess a formula for building S_k in terms of all previous $S_{k-1}, \ldots S_0$ and binomial symbols. You don't have to prove it, just guess it.

72. •• Open question. Use the building formulas to get answers for S_K. What is the highest k that you can do? Take your time. If you can get to S_{10} then your skill in algebra has progressed greatly. It may take a long time, but see how high you can go. Compare your work with these answers:

$$\begin{aligned}
S_5 &= \frac{n^2(n+1)^2(2n^2+2n-1)}{12} \\
S_6 &= \frac{n(n+1)(2n+1)(3n^4+6n^3-3n+1)}{42} \\
S_7 &= \frac{n^2(n+1)^2(3n^4+6n^3-n^2-4n+2)}{24} \\
S_8 &= \frac{n(n+1)(2n+1)(5n^6+15n^5+5n^4-15n^3-n^2+9n-3)}{90}.
\end{aligned}$$

Matrix sums

73. Divide m/d by Euclidean division. Find quotient q and remainder r. Write your answer in the form $m = qd + r$.

(a) 999/25.
(b) 200/17.
(c) 5/14.
(d) 19/13.

74. Construct a 5x5 matrix having 5^2 element consisting of consecutive integers beginning with 0. Like so:

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 \\ 20 & 21 & 22 & 23 & 24 \end{bmatrix}$$

Find the matrix sum S in two different ways. Is it the same in both cases?

75. Like problem **74** but starting at 1. Find the matrix sum in two different ways. Are they equal?

76. Construct a 5x5 matrix using consecutive numbers starting at 1. Now, swap (exchange) two rows. You can choose any two rows you like. Calculate the matrix sum. Do it again in a different way. Do you get the same sum?

77. Similar to problem **76** but now choose two *columns* and swap them. Do it two different ways. Do you get the same matrix sum?

78. The *transpose* of a matrix A is constructed by writing the rows of A as columns. We write A^T for the transpose of A. Here is a 3x3 example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Construct a 5x5 matrix with 5^2 consecutive integers beginning at 1. Find the matrix sum of the transpose of this matrix. Try it two different ways. Do you get the same sum?

79. Construct a 5x5 matrix using consecutive odd numbers beginning with 1. Find the matrix sum in two different ways. Do you get the same result both times?

80. Construct a 5x5 matrix using consecutive numbers congruent to 1 mod 7. Start with 1. Find the matrix sum in two different ways. Are the results equal?

81. • Like problem **80**, but use any modulus m and begin with any residue a. Fill a 5x5 matrix with the numbers

$$0m + a, \; 1m + a, \; 2m + a, \ldots$$

and find the matrix sum in two different ways. Are the results the same?

82. • Open question. Try combinations of the above ideas and any other ideas you have for constructing matrices of integers, then check if the matrix sum is always the same. Try to discover new ideas for which the matrix sum always gives you the same result no matter what elements you choose.

83. A 3D cube matrix can be represented by writing down its three faces. Let i be the index of rows, j be the index of columns and k be the face index. Each $k = 1, 2, 3$ corresponds to one of these in the cube:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \begin{bmatrix} 10 & 11 & 12 \\ 13 & 14 & 15 \\ 16 & 17 & 18 \end{bmatrix} \qquad \begin{bmatrix} 19 & 20 & 21 \\ 22 & 23 & 24 \\ 25 & 26 & 27 \end{bmatrix}$$

Construct a 4x4x4 cube this way and fill it with consecutive integers starting at 1. Calculate the matrix sum by choosing an element and eliminating three associated faces. Do it two different ways. Do you get the same result?

84. Find a formula for the matrix sum S for an $n \times n$ matrix with elements beginning at 0. Use the Euclidean division idea.

85. Find a formula for the matrix sum S for an $n \times n$ matrix with elements beginning at 1. Use the Euclidean division idea. You should get

$$S = \frac{n^3 + n}{2}.$$

86. • Prove that the matrix sum for the transpose of an $n \times n$ matrix beginning with 1 is also $(n^3 + n)/2$

87. •• Consider an $n \times n$ matrix of the type in problem **81**. Can you use the Euclidean division idea to find a formula for S?

88. •• Consider an $n \times n \times n$ version of the cube matrix in problem **83**. Fill it with integers starting at 1. Notice that that if you increase j (columns) by 1, the elements increase by 1. And when you increase i (rows) by 1, the elements increase by n. And when you increase k (faces) by 1, the elements increase by n^2. So a typical choice of row, column and face gives the element x

$$x = kn^2 + in + j + 1.$$

You can use this idea to prove that

$$S = \frac{n^4 + n}{2}.$$