M1 Training Problems

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1 Linear equations, sketches and exact plots

1. Solve $(4x-2)-(2x-1)=3x$. How many solutions does it have?

2. Solve $(3x-2)-(2x-1)=x$. How many solutions does it have?

3. Solve $(4x-1)-(2x-1)=2x$. How many solutions does it have?

4. Solve for x. $5x+6=3x+2$. How many solutions does it have?

5. Solve for x. $5x+2=5x+2$. How many solutions does it have?

6. Solve for x. $3x+1=3x-1$. How many solutions does it have?

7. Sketch freehand, no ruler: $y=x$, $y=-x$. Put them on the same axes. Remember to label everything.

8. Sketch freehand, no ruler: $y=2x$, $y=-2x$. Put them on the same axes.

9. Sketch freehand, no ruler.

$$y=\frac{x}{2},\quad y=-\frac{x}{2}.$$

Put them on the same axes.

10. Sketch freehand, no ruler. Label everything.

$$y=2x+1.$$

11. Sketch freehand, no ruler. Label everything.

$$y=-2x-1.$$

12. Sketch freehand, no ruler. Label everything.

$$y=-\frac{x}{2}+3.$$

13. Sketch freehand, no ruler. Label everything.

$$y=\frac{x}{2}-3.$$

14. Make an exact plot. Find the x and y intercepts. Show your work. Use a ruler.

$$y = 3x - 2.$$

15. Make an exact plot.

$$y = -\frac{x}{3} + 1.$$

16. Consider $y = 3x - 2$.
(a) Make a freehand sketch, no ruler.
(b) Make an exact plot, with a ruler. Find intercepts.

17. Consider the equation

$$2x + 3 = -\frac{x}{2} + 1.$$

(a) Solve for x by algebra. How many solutions does it have?
(b) Make exact plots of the left-hand side and right-hand side of the equation. Show where the solutions are.

18. Consider the equation

$$2x + 3 = 2x - 1.$$

(a) Solve for x by algebra. How many solutions does it have?
(b) Make exact plots of the left-hand side and right-hand side of the equation. Show where the solutions are. Make sure your plots match your algebra.

19. Consider the equation

$$-2x + 1 = -2x + 1.$$

(a) Solve for x by algebra. How many solutions does it have?
(b) Make exact plots of the left-hand side and right-hand side of the equation. Show where the solutions are. Make sure your plots tell the same story as your algebra.

20. Consider the equation

$$3x - 6 = x = \frac{x}{3} + 2.$$

Does this equation have a solution? Make exact plots of the left-hand side, the right-hand side, and the middle, all on the same axes. Show where the solution is, if there is one.

21. Consider the equation

$$\frac{x}{2} + 1 = 2x - 3 = -x + 4.$$

Use exact plots of left-hand side, right-hand side and middle to figure out if this has solutions. Show where the solutions are, if there are any.

22. Consider the three-way equation

$$x + 5 = 2x + 6 = 3x + 7.$$

Make exact plots of the LHS, RHS and middle. Show where the solutions are, if any.

2 One equation, many unknowns

23. The not-so-interesting cases for solutions of equations are when there are no solutions or anything is a solution. But what about the interesting ones? Draw the interesting cases for these equations:

(a) One unknown, $ax = b$.
(b) Two unknowns, $ax + by = c$.
(c) Three unknowns, $ax + by + cz = d$.

24. Fill in this table.

Equations and unknowns	What can happen?
One equation, one unknown	No solutions. Anything is a solution. All solutions are on one point.
One equation, two unknowns. $ax + by = c$	
One equation, three unknowns. $ax + by + cz = d$	

25. Consider the equation $ax = b$. By choosing numbers for a and b you can give examples for the different cases.

(a) Give an example where it has no solution.
(b) Give an example where anything is a solution.
(c) Give an example where there is one solution on one point.

26. Consider $ax + by = c$. Choose numbers for a, b, c and give examples for the following different cases:

(a) Give an example where it has no solution.
(b) Give an example where anything is a solution.
(c) Give an example where all solutions are on a line.

27. Consider $ax + by + cz = d$. Choose numbers for a, b, c, d and give examples for these different cases:

(a) Give an example where it has no solution.
(b) Give an example where anything is a solution.
(c) Give an example where all solutions are on a plane.

28. Consider the line $ax + by = c$. Find the x-intercept. Show how you found it.

29. Consider the line $ax + by = c$. Find the y-intercept. Show how.

30. Consider the line $ax + by = c$. Find the slope. Don't just write the answer. Show how.

31. Consider the equation $2x + y = 1$. Make an exact plot of this by the *abc* method. Find the x-intercept and y-intercept. Use a ruler. Label everything: axes, line, intercepts.

32. Make an exact plot of

$$-\frac{x}{2} + y = 3.$$

by the *abc* method.

33. Make an exact plot of $2y - x = -2$ by the *abc* method. Be careful.

34. Consider $3x - 2y = 1$.
- (a) Make an exact plot by the *abc* method. Use a ruler and label everything.
- (b) Find the slope.

35. Consider $\frac{x}{2} + \frac{y}{3} = 1$.
- (a) Make an exact plot by the *abc* method. Use a ruler. Label everything.
- (b) Find the slope.

3 Two equations, two unknowns

36. Give an example of how two lines can join to make a point. Draw.

37. Draw an example of two lines having no point in common.

38. Draw an example of two lines that join on an infinite number of points.

39. Draw an example of two planes having no points or lines in common.

40. Draw an example of two planes joining one one line.

41. Draw an example of two planes joining on an infinite number of different lines.

42. Draw an example of three planes joining to make a line.

43. Draw an example of three planes joining to make exactly one point.

44. Consider the system of equations:

$$\begin{aligned} x + 2y &= 1 \qquad (1) \\ x - y &= 2 \qquad (2) \end{aligned}$$

Plot (1) and (2) on the same axes by *abc* method and show where they are both true.

45. Consider the system

$$\begin{aligned} 2x - y &= 1 \qquad (1) \\ x + y &= 2 \qquad (2) \end{aligned}$$

- (a) Make exact plots of (1) and (2) by *abc* method and show where the solution is.

(b) Find the solution by algebra.
(c) Does your work in (a) match your work in (b)?

46. Consider the system

$$x - \frac{y}{3} = 1 \qquad (1)$$
$$\frac{x}{2} + y = 2 \qquad (2)$$

(a) Make exact plots of (1) and (2) by *abc* method and show where the solution is.
(b) Find the solution by algebra.
(c) Does your work in (a) match your work in (b)?

4 Cramer's method

47. Consider the system of two equations:

$$ax + by = e \qquad (1)$$
$$cx + dy = f \qquad (2)$$

(a) D is the system determinant. What happens when $D \neq 0$? Draw an example.
(b) What happens when $D = 0$? Draw examples. There are two possibilities.

48. Consider the system of two equations:

$$ax + by = e \qquad (1)$$
$$cx + dy = f \qquad (2)$$

(a) Let D_x be the x-determinant. What happens when $D_x = 0$? Where is the solution? Draw an example.
(b) Let D_y be the y-determinant. What happens when $D_y = 0$? Where will the solution be? Draw an example.
(c) What happens when both $D_x = 0$ and $D_y = 0$? Where is the solution? Draw an example.

49. Consider the system of two equations:

$$ax + by = e \qquad (1)$$
$$cx + dy = f \qquad (2)$$

(a) Put $x = \dfrac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$ and $y = \dfrac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$ into equation (1) and show that it is true.

(b) Put $x = \dfrac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$ and $y = \dfrac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$ into equation (2) and show that it is true.

50. Consider the system of two equations:

$$\begin{aligned} ax + by &= e \qquad (1) \\ cx + dy &= f \qquad (2) \end{aligned}$$

(a) Use algebra to solve for x and y. You should get

$$x = \frac{de - cf}{ad - bc}, \quad y = \frac{af - eb}{ad - bc}.$$

(b) Now write x and y using determinants. You have now discovered Cramer's method for yourself.

51. Consider the system of two equations in two unknowns:

$$\begin{aligned} x + 2y &= 3 \qquad (1) \\ 2x + 3y &= 2 \qquad (2) \end{aligned}$$

(a) Make exact plots of (1) and (2) by *abc* method and show where the solution is.
(b) Find the solution (x, y) by Cramer's method.

52.

$$\begin{aligned} -x + 5y &= 1 \qquad (1) \\ 3x - 2y &= 2 \qquad (2) \end{aligned}$$

(a) Make exact plots of (1) and (2) by *abc* method and show where the solution is.
(b) Find the solution (x, y) by Cramer's method.

5 Properties of determinants

53. Begin with $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. Swap the rows. What happens to the determinant?

54. Begin with $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$. Swap the columns. What happens to the determinant?

55. Begin with $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. Flip it over the long diagonal. What happens to the determinant?

56. Begin with $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. Flip it over the other long diagonal. What happens?

57. Start with $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. Add row 1 to row 2. What happens to the determinant?

58. Start with $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. Add row 2 to row 1. What happens to the determinant?

59. Start with $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. Add column 1 to column 2. What happens?

60. Start with $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. Add column 2 to column 1. Does the determinant change?

61. Start with $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. First add column 2 to column 1. Then add row 2 to row 1. Does the determinant change?

62. Start with $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. First swap rows. Then after that, swap the columns. Does the determinant change?

63. Start with $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. Multiply column 1 by -1. What happens to the determinant?

64. Start with $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. Multiply column 2 by -1. What happens to the determinant?

65. Start with $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. Multiply row 1 by -1. What happens to the determinant?

66. Start with $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. Multiply row 2 by -1. What happens to the determinant?

67. Start with $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. Multiply all the elements a, b, c and d by -1. What happens to the determinant?

68. Start with $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. Multiply row 1 by m and row 2 by n. What happens to the determinant? How does it change?

69. Start with $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. Multiply column 1 by m and column 2 by n. What happens to the determinant? How does it change?

70. Start with $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. Multiply row 1 by k and row 2 by $1/k$. What happens to the determinant? Does it change?

71. Start with $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. Multiply column 1 by k and column 2 by $1/k$. What happens to the determinant? Does it change?

6 Applications of determinants

72. Given that $\frac{a}{b} = \frac{c}{d}$ prove that $\frac{d}{b} = \frac{c}{a}$. Use determinant properties.

73. Given that $\frac{a}{b} = \frac{c}{d}$ prove that $\frac{a}{c} = \frac{b}{d}$. Use determinant properties.

74. Given that $\frac{a}{b} = \frac{c}{d}$ prove that $\frac{b}{a} = \frac{d}{c}$. Use determinant properties.

75. Given that $\frac{a}{b} = \frac{c}{d}$ prove that $\frac{a}{c} = \frac{a+b}{c+d}$. Use determinants.

76. Given that $\frac{a}{b} = \frac{c}{d}$ prove that $\frac{a+b}{b} = \frac{c+d}{d}$. Use determinants.

77. Given that $\frac{a}{b} = \frac{c}{d}$ prove that $\frac{a+b}{c+d} = \frac{b}{d}$. Use determinants.

78. Given that $\frac{a}{b} = \frac{c}{d}$ prove that $\frac{a}{b} = \frac{c-a}{d-b}$. Use determinants.

79. If $\frac{a}{b} = \frac{c}{d}$ prove that $\frac{a}{b-a} = \frac{c}{d-c}$.

80. Given that $\frac{a}{b} = \frac{c}{d}$ prove that $\frac{a}{b} = \frac{c+2a}{d+2b}$.

81. Given that $\frac{a}{b} = \frac{c}{d}$ prove that $\frac{a+c}{b+d} = \frac{a-c}{b-d}$.

82. Given that $\frac{a}{b} = \frac{c}{d}$ prove that $\frac{a+b}{a-b} = \frac{c+d}{c-d}$.

83. Plot these points $A(-3,-2)$, $B(6,-1)$, $C(1,4)$ and calculate the area of triangle ABC by using the determinant formula.

84. Write down the determinant area formula for a quadrilateral $ABCD$.

85. Write down the determinant area formula for a pentagon $ABCDE$.

86. Write down the determinant area formula for a hexagon $ABCDEF$.

87. Use the determinant area formula to show that the area of a straight line segment AB is zero.

88. Use the determinant area formula to show that the area of a point A is zero.

89. The points $A(-3,-2)$, $B(6,-1)$, $C(5,4)$ and $D(-1,2)$ make quadrilateral $ABCD$. Plot these points, draw the quadrilateral, and use the determinant area formula to find the area of $ABCD$.

90. Find the equation of the line going through the points $A(2,1)$, $B(6,3)$. Use determinant area method.

91. Find the equation of the line going through the points $A(-4,-3)$, $B(3,-1)$. Use determinant area method.

92. Consider points $A(-5,0)$, $B(0,2)$ and $A'(-10)$, $B'(0,-4)$.

(a) Plot these points and draw line (1) through A, B and line (2) through A', B'. Lines (1) and (2) are on point M. Show where is M.

(b) Use determinant area method to find the equation for line (1).
(c) Use determinant area method to find the equation for line (2).
(d) Use Cramer's method on the equations (1) and (2) to find solution M.

This is a hard problem, so here are the answers to each step. Line (1): $2x - 5y = -10$, line (2): $4x + y = -4$, point M: $(-15/11, 16/11)$.

93. Consider the points $A(-4,0)$, $A'(0,4)$, $B(-1,0)$, $B'(0,-1)$, $C(-3,0)$, $C'(4,1)$, $D(2,0)$, $D'(0,5)$. Each pair of points makes a line.

(a) Plot points A, A', B, B', C, C', D, D'.
(b) Draw lines AA', BB', CC', DD'.
(c) Show where lines AA' and BB' meet. Call this point M. Draw it.
(d) Show where lines CC' and DD' meet. Call this point N. Draw it.
(e) Draw the line through points M and N. Our job is to find the equation of this line. We will use only determinant-area method and Cramer's method.
(f) Use determinant area method to find equations for lines AA' and BB'. Call them equations (1) and (2).
(g) Use Cramer's method to find point M from equations (1) and (2).
(h) Use determinant area method to find equations for lines CC' and DD'. Call them equations (3) and (4).
(i) Use Cramer's method to find the point N from equations (3) and (4).
(j) Finally, use determinant area method to find the line that goes through point M and N.

Congratulations if you did all this. Solution: line AA': $x - y = -4$, line BB': $x + y = -1$, line CC': $x - 7y = -3$, line DD': $5x + 2y = 10$. Point $M(-5/2, 3/2)$. Point $N(64/37, 25/37)$. Line MN: $61x + 313y = 317$.