SME M1 Training Problems

Ted Szylowiec

tedszy@gmail.com

Problems that may be harder than usual (or more advanced) are marked with a dot •. Problems that are possibly even harder (and even more advanced) are marked with two bullets ••. All problems are worth doing! Try to do all of them. Most don't take very long. Some of the problems include answers.

Solving for x and N-way fractions

1. Solve $(8x-3)-(7x-2)=1$.

Answer: $x=2$.

2. Solve $(5x-4)-(4x-3)-(3x-2)=2x-1$.

Answer: $x=1/2$.

3. Solve for x:

$$(1-2x)-(2-3x)-(3-4x)-(4-5x)-(5-6x)=10.$$

Answer: $x=23/16$.

4. Prove this two-way formula

$$\frac{1}{(n-2)(n-1)}=\frac{1}{n-2}-\frac{1}{n-1}.$$

by taking the right-hand side, doing some algebra, and showing that it is equal to the left-hand side.

5. Use a two-way formula similar to the one in problem **4**. Solve for x.

$$\frac{x}{10\cdot 11}+\frac{x}{11\cdot 12}+\frac{x}{12\cdot 13}+\frac{x}{13\cdot 14}+\frac{x}{14\cdot 15}=1.$$

Answer: $x=30$.

6. Factor the denominators into two parts and use a two-way formula like in problem **4**. Solve for x.

$$\frac{x}{90}+\frac{x}{110}+\frac{x}{132}+\frac{x}{156}=\frac{1}{13}.$$

Answer: $x=9/4$.

7. Use a two-way fraction formula like in problem **4** and find the sum.

$$\frac{1}{10\cdot 11}+\frac{1}{11\cdot 12}+\cdots+\frac{1}{100\cdot 101}$$

Answer: $91/1010$.

8. Prove that
$$\frac{1}{(n+1)(n+2)(n+3)} = \frac{1}{2}\left(\frac{1}{(n+1)(n+2)} - \frac{1}{(n+2)(n+3)}\right).$$

9. Use a three-way formula and solve for x:
$$\frac{x}{10\cdot 11\cdot 12} + \frac{x}{11\cdot 12\cdot 13} + \frac{x}{12\cdot 13\cdot 14} + \frac{x}{13\cdot 14\cdot 15} = 1.$$

Answer: $x = 462$.

10. Factor the denominators into three-way products. Use a three-way formula. Solve for x.
$$\frac{x}{120} + \frac{x}{210} + \frac{x}{336} + \frac{x}{504} = 13.$$

Answer: $x = 720$.

11. Use a three-way formula to find the sum. You may need to use a calculator.
$$\frac{1}{10\cdot 11\cdot 12} + \frac{1}{11\cdot 12\cdot 13} + \cdots + \frac{1}{99\cdot 100\cdot 101}.$$

Answer: 999/222200.

12. Prove that
$$\frac{1}{(n-3)(n-2)(n-1)n} = \frac{1}{3}\left(\frac{1}{(n-3)(n-2)(n-1)} - \frac{1}{(n-2)(n-1)n}\right).$$

13. Use a four-way formula to solve for x.
$$\frac{x}{2\cdot 3\cdot 4\cdot 5} + \frac{x}{3\cdot 4\cdot 5\cdot 6} + \frac{x}{4\cdot 5\cdot 6\cdot 7} + \frac{x}{5\cdot 6\cdot 7\cdot 8} = 13.$$

Answer: $x = 1008$.

14. Factor the denominators into four-way products and use a four-way formula to solve for x.
$$\frac{x}{5040} + \frac{x}{7920} + \frac{x}{11880} = 17.$$

Answer: $x = 41580$.

15. Use a four-way formula to find the sum. You may need to use a calculator.
$$\frac{1}{10\cdot 11\cdot 12\cdot 13} + \frac{1}{11\cdot 12\cdot 13\cdot 14} + \cdots + \frac{1}{98\cdot 99\cdot 100\cdot 101}.$$

Answer: 1513/5999400.

16. • Study the patterns in these two, three and four-way formulas:
$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$
$$\frac{1}{n(n+1)(n+2)} = \frac{1}{2}\left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right)$$
$$\frac{1}{n(n+1)(n+2)(n+3)} = \frac{1}{3}\left(\frac{1}{n(n+1)(n+2)} - \frac{1}{(n+1)(n+2)(n+3)}\right).$$

Now take a guess for a five-way formula beginning like this:

$$\frac{1}{n(n+1)(n+2)(n+3)(n+4)} = \cdots$$

17. • Prove your guess for the five-way formula in problem **16**.

18. •• Take a guess at a k-way formula. It begins like this on the left-hand side:

$$\frac{1}{n(n+1)(n+2)\cdots(n+k-1)} = \cdots .$$

19. •• Prove the k-way formula that you guessed in problem **18**. You can use the three dots (ellipsis) notation $\cdots$ wherever you need it.

20. • Prove these. Use two, three and four-way formulas to change the left-hand side into the right-hand side.

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(n-1)n} = 1 - \frac{1}{n}$$

$$\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \cdots + \frac{1}{(n-2)(n-1)n} = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{(n-1)n}\right)$$

$$\frac{1}{1\cdot 2\cdot 3\cdot 4\cdot} + \frac{1}{2\cdot 3\cdot 4\cdot 5} + \cdots + \frac{1}{(n-3)(n-2)(n-1)n} = \frac{1}{3}\left(\frac{1}{6} - \frac{1}{(n-2)(n-1)n}\right).$$

21. • Study the patterns in problem **20**. Guess the next one. It will be a sum of five-way terms.

22. • Prove answer to problem **21** by using the formula that you guessed in **16**.

23. •• Examine the patterns in the sums of problems **20** and **21** and take a guess for the k-way sum.

24. • • • Use your result from problem **18** to prove your answer to problem **23**.

Sets and intervals

25. Figure them out...

(a) $\{\varnothing\} \cap \varnothing$.
(b) $\{\varnothing\} \cap \{\varnothing\}$.
(c) $\{\varnothing\} \cup \varnothing$.
(d) $\{\varnothing\} \cup \{\varnothing\}$.

26. Figure these out...

(a) $\{1, \varnothing\} \cap \varnothing$.
(b) $\{1, \varnothing\} \cap \{\varnothing\}$.
(c) $\{1, \varnothing\} \cup \varnothing$.
(d) $\{1, \varnothing\} \cup \{\varnothing\}$.

27. Let $A = \{2, 5, 9, 11, 13\}$ and $B = \{1, 9, 11, 14, 21\}$. Figure out $A \cup B$ and $A \cap B$.

28. Let $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 4, 6, 7, 8\}$ and $C = \{2, 7, 8, 10, 11\}$. Find the following sets

(a) $(A \cup B) \cap C$.
(b) $A \cup (B \cap C)$.
(c) $(A \cap B) \cup (B \cap C)$.
(d) $(A \cup B) \cap (B \cup C)$.

29. Draw these intervals on the real line.

(a) $(0, 2) \cup (3, 5)$.
(b) $[-2, 1] \cup [0, 2]$.
(c) $(-\infty, 1) \cup (-1, \infty)$.
(d) $[-1, \infty) \cap (-\infty, 1]$.
(e) $[-2, 5] \cap [1, 10]$.
(f) $(-2, 5) \cap (1, 10)$.

30. Draw these intervals on the real line.

(a) $(-\infty, \infty)$.
(b) $[1, 1]$.
(c) $(1, 1)$.
(d) $[1, 2) \cap (2, 3]$.
(e) $[1, 2] \cap [2, 3]$.

31. Write as inequalities.

(a) $(-\infty, 2]$.
(b) $(2, \infty)$.
(c) $(1, 5)$.
(d) $[1, 5]$.

32. What do you get?

(a) $\mathbb{Q} \cap \mathbb{N}$.
(b) $\mathbb{Z} \cup \mathbb{R}$.
(c) $\mathbb{R} \cap \mathbb{Q}$.
(d) $\mathbb{Z} \cap [-5, 5]$.
(e) $\mathbb{Z} \cap (-5, 5)$.
(f) $\mathbb{N} \cap (-5, 5)$.

33. • Let $\mathbb{M} = \{-n, n \in \mathbb{N}\}$. Is $\mathbb{M} \cup \mathbb{N}$ the same set as $\mathbb{Z}$? If not, what is the difference? What is $\mathbb{M} \cap \mathbb{N}$?

34. • Let A be the set of all integers divisible by 5. Let B be the set of all integers divisible by 3. Figure out the following...

(a) $A \cup B$.
(b) $A \cap B$.

Solving equalities and inequalities

35. Solve these inequalites. Draw the solutions on the real line.
(a) $x < 5$.
(b) $x < 5$ and $x > -1$.
(c) $x < 5$ or $x > -1$.
(d) $x \leqslant 5$ and $x \geqslant -5$.
(e) $x \leqslant 5$ or $x \geqslant -5$.

36. Solve for x. Express your answer using sets and also drawings on the real line.
(a) $(5x + 1) = 6x - 5 - (x - 5)$.
(b) $(5x + 1) = 6x - 5 - (x - 6)$.

---

37. Solve for x.

$$ax = 0.$$

Where a may be any real number. Give analysis of cases. Use proper mathematical language (*if... then...*) and give a proper conclusion. Do so for all problems that require analysis of cases. Use set theory to describe your answers: $\mathbb{R}$, $(-\infty, \infty)$, $\varnothing$, etc.

38. Solve for x, where a may be any real number: $ax = a$.

39. Solve for x, where a can be anything: $ax = x$.

40. Solve for x: $ax = a + 1$.

41. Solve for x: $ax = a - 1$.

42. Solve for x: $ax = x + 1$.

43. Solve for x: $ax = x - 1$.

44. Solve for x: $a^2 x = a + a^2$,

45. Solve for x: $(a - 1)x = a$.

46. Solve. $(a + 1)x = 0$. Remember, a can be any real number.

47. Solve for x: $(a - 1)x = a^2 - 1$.

48. Solve. $(a + 3)x = a^2 - 9$.

49. Solve for x: $(a^2 - 4) = a^3 + 8$.

50. Solve. $(a^2 - 5) = a^3 - 125$.

51. Solve for x: $(a - 2)(a + 3)x = 1$.

52. Solve. $(a + 2)(a - 2)x = a^2 - 4$.

53. Solve. $(a + 2)(a - 3) = a^2 - 9$.

54. Solve for x. $x + 3 \leqslant -1$. Draw the solution on the real line. Give the solution as an interval.

55. Solve. Draw. Give interval: $x > x + 1$.

56. Solve. Draw. Give interval: $x \leqslant x + 1$.

57. Solve for x: $x - 3 \leqslant x + 2$. Draw solution on real line. Give interval.

58. Solve for x: $x - 3 \geqslant x + 3$. Draw solution on real line. Give interval.

59. Solve for x: $2x - 1 > 5$.

60. Solve for x: $-3x + 1 < 10$.

61. Solve. $2x + 4 < 5x - 1$.

62. Show these on the real line: 0, 2, -2, $1/2$, $-1/2$.

63. Show on real line: $\sqrt{2}$, $1/2$, 2, -2, $1/\sqrt{2}$, $-\sqrt{2}$, $-1/\sqrt{2}$, $-1/2$, -2.

64. Show on real line: 1, 2, 3, $\sqrt{2}$, $1/\sqrt{2}$, $\sqrt{3}$, $1/\sqrt{3}$.

65. Solve for x: $\sqrt{2}x + 1 > \sqrt{3}x - 2$. Draw solution on real line. Give interval for solution.

66. Solve. $\sqrt{7}x - 1 \leqslant \sqrt{5}x + 2$.

67. Solve. Draw. Give interval. $\sqrt{3}x + 1 \geqslant 2x - 3$.

68. Solve. Draw. Give interval. $3x + 2 < \sqrt{5}x + 4$.

69. Let $a > 0$. Show 0 and $1/a$ on the real line.

70. Let $a < 0$. Show 0 and $1/a$ on the real line.

71. Let $a > 0$. Show 0 and a on the real line.

72. Let $a < 0$. Show 0 and a on the real line.

73. let $a > 0$. Draw 0 and a^2 on the real line.

74. Let $a < 0$. Draw 0 and a^2 on the real line.

75. Let $a < 0$. Draw 0 and $1/a^2$ on the real line.

76. Solve for x: $ax > 1$, where a can be any number. Analyze all cases. Write a conclusion. Use mathematical language (*if... then....*) Explain what you are doing. Draw the solutions on the real line. Give solutions as intervals.

77. Solve for x, draw, give intervals: $ax < 1$.

78. Solve for x, draw, give intervals: $a^2x \geqslant 0$.

79. Solve for x, draw, give intervals: $a^3x \leqslant 0$.

80. Solve, draw, give intervals. $(a-3)^2x > 1$.

81. • Solve, draw, give intervals. $(a-3)^3x < 1$.

Systems of inequalities

82. Show these numbers on the real line: π. $\pi/2$, 1, 2, 3, 4, 10, π^2.

83. Show these numbers on the real line: 0, 1, 2, 3, 4, -1, -2, -3, -4, π, $-\pi$, $\pi/2$, $-\pi/2$.

84. Show these numbers on the real line: 0, 1, 2, 3, 4, $\sqrt{2}$, $\sqrt{3}$ $\sqrt{5}$, $\sqrt{7}$, $\sqrt{10}$.

85. Solve this system, draw the solution on the real line and give the solution as intervals.

$$x < 3; \quad -3x < 9.$$

86. Solve. Draw. Give intervals. Remember that the solution of a system of two equations makes both of them true.

$$2x \geqslant 4; \quad x \geqslant 2.$$

87. Solve for x. Draw. Give intervals.

$$-2x = \pi; \quad -x < 1.8$$

88. Solve for x. Draw. Give intervals.

$$\pi - 2x < 0; \quad -\sqrt{3} - x > 0.$$

89. Solve. Draw. Write solution as a set.

$$6 - 2x \geqslant 10; \quad x \in \mathbb{N}.$$

90. Solve. Draw. Give solution as a set.

$$2x + 10 > -10; \quad 10 - 2x > 5; \quad x \in \mathbb{Z}.$$

91. Solve. Draw. Give solution as a set.

$$x - 2\pi < 3x; \quad -x > -\sqrt{5}; \quad x \in \mathbb{Z}.$$

Absolute value

92. Fill in the table.

Transformation	What does it do?
$y \to y - 1$	
$x \to x - 1$	
$y \to y + 1$	
$x \to x + 1$	

93. Plot the straight line $y = x + 3$. Draw before and after pictures. Start with the basic straight line $y = x$. Do the correct transformation. Draw the result. Label the axes and the lines.

94. Plot $y + 1 = x - 2$ by starting with the line $y = x$ and doing two transformations. Next, plot the line $y = x - 3$ by starting with $y = x$ and doing one transformation. Are they the same?

95. Plot these lines on the same axes by transforming x.

$$y = x - 3; \quad y = x - 1; \quad y = x; \quad y = x + 1; \quad y = x + 3.$$

96. Plot these lines on the same axes by transforming y.

$$y - 3 = x; \quad y - 1 = x; \quad y = x; \quad y + 1 = x; \quad y + 3 = x.$$

97. Plot $y = |x + 2|$ by transforming x.

98. Plot $y - 1 = |x + 2|$ by transforming x and y.

99. Plot $y = |x - 1| + 2$ by transforming x and y.

100. Plot the lines $y = 2$ and $y = -2$ on the same axes.

101. Solve $|x| = 2$. First plot $y = |x|$ and $y = 2$ on the same axes. Use your drawing to figure out how many solutions there should be. Next, find the solutions by using the definition of absolute value.

102. Solve $|x| = 0$. First plot $y = |x|$ and $y = 0$ on the same axes. Find the solutions by using the definition of absolute value (*if... then...*).

103. Solve $|x| = -2$. First plot $y = |x|$ and $y = 0$ on the same axes. Show that there are no solutions by using the definition of absolute value.

104. •. Solve for x:

$$|x| = a$$

where a can be any number. You will have to analyze different cases for a. For each case draw plots and then find solutions for x if there are any.

105. Solve for x:

$$|x - 3| = 1.$$

Draw plots of $y = |x - 3|$ and $y = 1$. Solve using the definition of absolute value.

106. Solve for x:

$$|x + 2| = 3.$$

Draw plots of $y = |x + 2|$ and $y = 3$. Solve using the definition of absolute value.

107. •. Solve for x:

$$|x - 1| = a$$

where a can be any number. Analyze different cases for a. For each case draw plots for $y = |x - 1|$ and $y = a$. Explain how many solutions there should be. Find them using the definition of absolute value.

108. Plot these on the same axes.

$$y = -x, \quad y = -(x + 1), \quad y = -(x - 1).$$

109. Use plots to show that
$$|x + a| = 1$$
always has two solutions no matter whether $a > 0$, $a = 0$ or $a < 0$. Use logical analysis to find the two solutions.

110. Use plots and show that $|x - a| = 3$ always has two solutions no matter what a is. Find the two solutions using logical analysis.

111. Plot these on the same axes.
$$y = -x, \quad y = -2 - x, \quad y = 2 - x.$$

112. Plot and solve.
$$|x + 1| = -(x - 1).$$

113. Plot and solve. Make sure your logical analysis matches what you see in the plots.
$$|x - 1| = x + 1$$

114. Use plots to determine how many solutions this has and find the solutions using logical analysis.
$$|x + 2| = x + 2$$

115. Plot and solve using logical analysis.
$$|x - 2| = 2 - x.$$

Remember: your logical analysis must match what you see on your plots. Let the plots guide you to doing it correctly.