Polygonal Numbers

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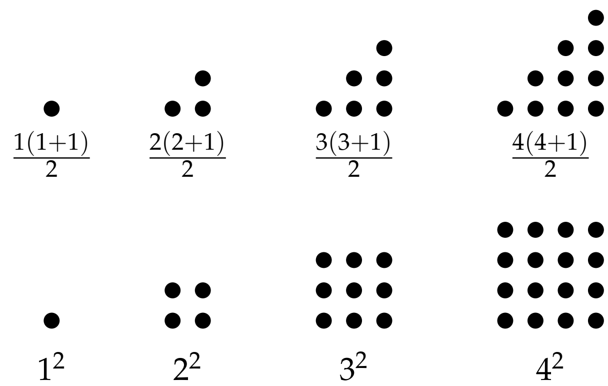
1 Square and triangular numbers

If n dots can be arranged in a regular shape then n is a polygonal number. Triangular numbers and square numbers are two familiar kinds of polygonal numbers. A question immediately comes to mind: is it possible for a number to be both square *and* triangular? This question is surprisingly deep. A full study of it involves a large amount of mathematical machinery: quadratic irrationals, matricies, recursion, continued fractions, the Euclidean algorithm, Diophantine equations and more.

A number n is a triangular number if n dots can be arranged in a triangle. If n is the kth triangular number, then

$$n = \frac{k(k+1)}{2}.$$

If n is a square number then n dots can be arranged in a square. The kth square number is $n = k^2$. Here are the first few triangular and square numbers, along with their dot diagrams:



From looking at the dots, the number $n = 1 = 1^2 = 1(1+1)/2$ is both triangular and square. So, at least one such number exists. It can be obtained in a different way. Suppose we ask: when is the kth triangluar number equal to the kth square number? To answer this, we must find the the solutions of

$$\frac{k(k+1)}{2} = k^2. \tag{1}$$

This is a simple quadratic equation with solutions $k = 0$ and $k = 1$. It tells us that the 0th triangular number is also the 0th square number, and the 1st triangular number is also the 1st square number. There is no other k such that the kth triangular number is also the kth square number.

Why not? It's easy see why by looking at the dot diagrams. For every $k > 1$, the kth triangular number is smaller than the kth square number. They can never be equal when

$k > 1$. This give a clue about how to proceed: for $k > 1$, the kth triangular number can only be equal to a square integer j^2 if $j < k$. Replace k in the right-hand side of (1) with something that can be made smaller: $k - h$, where h is some integer:

$$\frac{k(k+1)}{2} = (k-h)^2. \tag{2}$$

When $h = 0$ we recover the solutions that we already found: $k = 0$ and $k = 1$, corresponding to the square-triangular numbers $n = 0$ and $n = 1$. In principle we can find all solutions by working our way through all possible values of h:

$$h = 0, 1, 2, 3, \ldots.$$

Putting $h = 1$ in (2) gives the quadratic equation

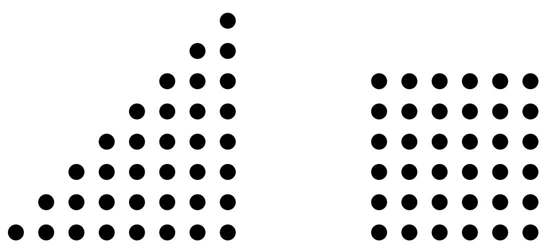
$$k^2 - 5k + 2 = 0.$$

Does this have any integer solutions? If the discriminant of a quadratic polynomial is a perfect square, then it might have integer roots. Here the discriminant of the right-hand side is 28, so no, it doesn't. We can interpret this negative result like so: there is no k such that the kth triangular number is equal to the $(k-1)$th square number.

Now put $h = 2$ in (2),

$$k^2 - 9k + 8 = 0.$$

The polynomial has discriminant 49, so this equation possibly does have integer solutions. We find them, either by factoring or by the quadratic formula: $k = 1$ and $k = 8$. The first solution says that the triangluar number 1 is equal to the square number $(1-2)^2 = (-1)^2$. This is true, but we are more interested in squares with positive sides. Nevertheless, it is good that we are getting some solutions we didn't expect. The second solution, $k = 8$, corresponds to the 8th triangular number, $8(8+1)/2 = 36$, which happens to be the 6th square number, $6^2 = 36$. Have a look at them side-by-side. Since $h = 2$, the sides of the square are two dots smaller than the sides of the triangle.



$\frac{8(8+1)}{2} = 36$ $(8-2)^2 = 6^2 = 36$

For $h = 3$ we have the equation

$$k^2 - 13k + 18 = 0.$$

The discriminant is 97, so there are no integer solutions. We keep working like this, by brute force, using the discriminant method to judge whether there are integer solutions, until we get to $h = 14$ and its corresponding equation:

$$k^2 - 57k + 392 = 0.$$

The discriminant of the right hand side is 1681, which is a perfect square, 41^2, so the equation possibly has integer roots. Using the quadratic formula we see that it does: $k = 8$ and $k = 49$. The first solution says that the 8th triangular number, 36, is equal to a square number $(8 - 14)^2 = (-6)^2 = 36$. The second solution, $k = 49$, corresponds to the triangular number 1225, which is also equal to the square $(49 - 14)^2 = 35^2$.

It may take a long time to discover the next triangular-square number by brute-force hand calculations. We could employ some tricks to speed up the analysis of each polynomial, but it would still be brute force. We need a more sophisticated approach.

2 Pell's equation

The thing to do is to study the nature of equation (2). It is a quadratic equation in two variables, k and h, with the additional condition that h and k are integers. This is known as a Diophantine equation.

There are many kinds of Diophantine equations, each having their own body of theory and techniques for finding solutions. For example, this is a linear Diophantine equation in two integer unknowns, x and y:

$$mx + ny = 1.$$

There are well-developed theories for solving this integer equation, based on the Euclidean algorithm or on continued fractions or on other ideas. When restricted to integers, the Pythagorean equation

$$x^2 + y^2 = z^2$$

is a second-order Diophantine equation in three integer unknowns x, y, z. Another well-known Diophantine equation is Pell's equation:

$$x^2 - Ay^2 = 1,$$

where x and y are integer unknowns and A is an integer that is not a perfect square. Pell's equation has a long and interesting history, and some of the greatest mathematicians, like Leonhard Euler, contributed to the theory of it.

Can (2) be fit into a known type of Diophantine equation for which there are theories and techniques for finding solutions? It turns out that, with a bit of algebra, equation (2) can be rewritten as an instance of Pell's equation.

Multiply equation (2) by 8:

$$4k(k+1) = 4k^2 + 4k = 8(k-h)^2.$$

Add 1 to both sides to complete the square,

$$\begin{aligned} 4k^2 + 4k + 1 &= 8(k-h)^2 + 1 \\ (2k+1)^2 &= 8(k-h)^2 + 1 \\ (2k+1)^2 - 8(k-h)^2 &= 1 \\ (2k+1)^2 - 2(2(k-h))^2 &= 1. \end{aligned}$$

And now, if we let $x = 2k + 1$ and $y = 2(k - h)$ we get Pell's equation with $A = 2$,

$$x^2 - 2y^2 = 1. \tag{3}$$

There are several theoretical methods available for studying this Diophantine equation. Continued fractions, quadratic irrationals, matricies, recursion, etc. We will borrow some ideas from the theory of quadratic irrationals. Just enough to find a few more triangular-square numbers.

3 Quadratic irrationals

A quadratic irrational is a number of the form

$$x + y\sqrt{A}$$

where x and y are integers and A is not a perfect square. If A is not a perfect square, then $y\sqrt{A}$ is an irrational number. The number $x + y\sqrt{A}$ consists of two parts: an integer part x and an irrational part $y\sqrt{A}$. Quadratic irrationals have some important properties: if two quadratic irrationals are equal, then their integer parts are equal, and their irrational parts are equal. In other words, if

$$x_1 + y_1\sqrt{A} = x_2 + y_2\sqrt{A}$$

then

$$x_1 = x_2 \quad \text{and} \quad y_1 = y_2.$$

With respect to this property, quadratic irrationals behave like vectors or complex numbers. When two quadratic irrationals of the same type (having the same A) are multiplied together, the result is another quadratic irrational of the same type:

$$\begin{aligned}(x + y\sqrt{2})(x' + y'\sqrt{2}) &= (xx' + 2yy') + (xy' + yx')\sqrt{2} \\ &= x'' + y''\sqrt{2}.\end{aligned}$$

It follows that when a quadratic irrational is raised to any power n, the result is another quadratic irrational of the same type. It's interesting to prove this using the binomial theorem.

$$\begin{aligned}(x + y\sqrt{A})^n &= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y\sqrt{A} + \cdots + \binom{n}{n}(y\sqrt{A})^n \\ &= x_n + y_n\sqrt{A}. \end{aligned} \tag{4}$$

Every term that has an even power of $\sqrt{A}$ is an integer. Every term that has an odd power of $\sqrt{A}$ is an integer times the irrational $\sqrt{A}$. We have collected all the integer terms into x_n and all the irrational terms into $y_n\sqrt{A}$. The result is a quadratic irrational of the same type as $x + y\sqrt{A}$.

The equation $x^2 - 2y^2 = 1$ can be factored into quadratic irrationals:

$$(x + y\sqrt{2})(x - y\sqrt{2}) = 1.$$

Notice that if x, y is a solution, then $x, -y$ is also a solution. This corresponds to the solutions we saw earlier having negative sides. Now raise both sides to the power of n,

$$(x + y\sqrt{2})^n(x - y\sqrt{2})^n = 1.$$

By (4) this can be written as

$$(x_n + y_n\sqrt{2})(x_n - y_n\sqrt{2}) = 1.$$

In other words: if x, y is a solution, then x_n, y_n is also a solution. This gives us a beautiful way of generating new solutions from old solutions: just take a known solution and write it as a quadratic irrational, then raise it to some power.

All Pell equations have a trivial solution, $x = 1$, $y = 0$. But when $y = 0$, we cannot make a quadratic irrational. So our technique will not work if we try to begin with the trivial solution. We must start with the first non-trivial solution.

Recall that

$$x = 2k + 1, \quad y = 2(k - h).$$

The trivial solution $x = 1$, $y = 0$ corresponds to $k = 0$, $h = 0$. The first non-trivial solution that we found was $k = 1$, $h = 0$. In terms of x and y, this is

$$x = 2(1) + 1 = 3, \quad y = 2(1 - 0) = 2.$$

Both x and y are non-zero, so we can begin with the solution $x = 3$, $y = 2$. Write it as a quadratic irrational:

$$3 + 2\sqrt{2}.$$

Now, raise it to the power of 2:

$$\left(3 + 2\sqrt{2}\right)^2 = 17 + 12\sqrt{2}.$$

We get a new solution, $x = 17$, $y = 12$. This corresponds to $k = 8$, $h = 2$, which was the first solution we found by brute force.

Raise the quadratic irrational to the 3rd power,

$$\left(3 + 2\sqrt{2}\right)^3 = 99 + 70\sqrt{2}.$$

This solution, $x = 99$, $y = 70$, corresponds to $k = 49$, $h = 14$, which is the triangular-square number $49(49 + 1)/2 = 1225 = 35^2$.

We are ready to go one step beyond. Raise the quadratic irrational to the 4th power:

$$\left(3 + 2\sqrt{2}\right)^4 = 577 + 408\sqrt{2}.$$

The solution $x = 577$, $y = 408$ corresponds to $k = 288$, $h = 84$. This gives the triangular-square number $288(288 + 1)/2 = 41616 = 204^2$. The sides of the square are 84 dots smaller than the sides of the triangle. Going yet farther, we arrange our computations in a table:

$(x + y\sqrt{2})^n$	Expansion	k	h	Triangular-square
$(3 + 2\sqrt{2})^1$	$3 + 2\sqrt{2}$	$k = 1$	$h = 0$	$1 = 1^2$
$(3 + 2\sqrt{2})^2$	$17 + 12\sqrt{2}$	$k = 8$	$h = 2$	$36 = 6^2$
$(3 + 2\sqrt{2})^3$	$99 + 70\sqrt{2}$	$k = 49$	$h = 14$	$1225 = 35^2$
$(3 + 2\sqrt{2})^4$	$577 + 408\sqrt{2}$	$k = 288$	$h = 84$	$41616 = 204^2$
$(3 + 2\sqrt{2})^5$	$2378 + 2263\sqrt{2}$	$k = 1681$	$h = 482$	$1413721 = 1189^2$
$(3 + 2\sqrt{2})^6$	$19601 + 13860\sqrt{2}$	$k = 9800$	$h = 2870$	$48024900 = 6930^2$

We found more solutions, but there are lingering questions. Does the quadratic irrational technique generate *all* solutions, or just some of them? Also, it seems we were lucky to find the first non-trivial solution easily. What would we do if our Pell equation was not so simple? For instance, the first non-trivial solution of $x^2 - 61y^2 = 1$ is the astoninshing pair

$$x = 1766319049, \quad y = 226153980.$$

Do all Pell equations have non-trivial solutions? Is there a technique for finding the first non-trivial solution? If you are a student who needs a mathematics topic for a self-study project, these questions would make a fine beginning.

Exercises

1. Prove that the sum of two consecutive triangular numbers is a square number.

2. Verify that $\sqrt{2} + \sqrt{3}$ is a solution of $x^4 - 10x^2 + 1 = 0$.

3. Verify that $x = 649$, $y = 180$ is a solution of $x^2 - 13y^2 = 1$.

4. The first non-trivial solution of $x^2 - 3y^2 = 1$ is $x = 2$, $y = 1$. Find four more solutions using the technique of quadratic irrationals.

5. The first non-trivial solution of $x^2 - 15y^2 = 1$ is $x = 4$, $y = 1$. Find two more solutions.

6. Use trial-and-error to find the first non-trivial solution of $x^2 - 6y^2 = 1$.

7. Find one more triangluar-square number and add your computations to the table.