Patterns in Pascal's Triangle: 92 Problems

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1 Sets and Subsets

1. How many subsets does a set of n elements have? Form a conjecture.

Find a pattern by enumerating the subsets of set S for various sizes of S. Start with size $n = 0$, which corresponds to the empty set. The empty set has only one subset.

n	S	subsets
0	$\{\}$	$\underline{\{\}}$
1	$\{a\}$	$\underline{\{\}}, \{a\}$
2	$\{a,b\}$	$\underline{\{\}, \{a\}}, \{b\}, \{a,b\}$.

Notice the pattern. The $n = 2$ problem contains the subsets of the $n = 1$ problem, and so on. Also, notice that each step has twice as many subsets as the previous step. Let $s(n)$ be the number of subsets of a set of size n. We have:

$$s(0) = 1$$
$$s(1) = s(0) + s(0)$$
$$s(2) = s(1) + s(1)$$

and we conjecture that

$$s(n) = 2\,s(n-1)$$
$$s(0) = 1.$$

The solution to this recursion is the sequence $1, 2, 4, 8, \ldots$. Or, in closed form:

$$s(n) = 2^n.$$

This is a conjecture, not a complete proof. We have assumed that the pattern is always true for all n. To complete the proof we would have to explain why it is true.

2. Extend the table in **1** to $n = 3$.

3. Extend the table in **1** to $n = 4$.

4. Prove the conjecture in **1** by means of a combinatorial argument.

We illustrate the argument with a small set of three elements. One of the three elements will be considered special, so give it a special label: x:

$$S = \{a, b, x\}.$$

Compare the subsets that do not contain x with the subsets that do contain x. There is a one-to-one correspondence. Every subset that contains x can be made by adding an x to a particular subset that does not contain x.

without x		with x
$\{\}$	$\longleftrightarrow$	$\{x\}$
$\{a\}$	$\longleftrightarrow$	$\{a, x\}$
$\{b\}$	$\longleftrightarrow$	$\{b, x\}$
$\{a, b\}$	$\longleftrightarrow$	$\{a, b, x\}$

The subsets of $\{a, b, x\}$ without x are precisely the subsets of $\{a, b\}$. There are an equal number of subsets having x as there are subsets without x. From the data in the table we can write:

$$\begin{aligned} s(3) &= s(2) + s(2) \\ &= 2\,s(2). \end{aligned}$$

This argument can be generalized to a set of size n where one of them is chosen to be special. We would then have a table of two columns with $s(n-1)$ rows. Again we obtain a recursion relationship:

$$\begin{aligned} s(n) &= s(n-1) + s(n-1) \\ &= 2\,s(n-1) \end{aligned}$$

with $s(0) = 1$. The solution to this recursion relationship is the sequence $1, 2, 4, 8, \ldots$ or $s(n) = 2^n$.

5. Make the table as in **4** for the set $S = \{a, b, c, x\}$.

6. Prove the result in **1** by counting subsets according to their length and then applying the binomial theorem.

Take $n = 4$, $S = \{a, b, c, d\}$ as an illustration. The subsets can range in size from 0 elements to 4 elements. Arrange them in a table:

size	subsets	number of subsets
0	$\{\}$	1
1	$\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$	4
2	$\{a,b\}$, $\{a,c\}$, $\{a,d\}$, $\{b,c\}$, $\{b,d\}$, $\{c,d\}$	6
3	$\{a,b,c\}$, $\{a,b,d\}$, $\{a,c,d\}$, $\{b,c,d\}$	4
4	$\{a,b,c,d\}$	1

This immediately suggests Pascal's triangle. It's clear why. There is 1 way to choose nothing from S. There are $\binom{4}{1}$ ways of choosing 1 object from S, $\binom{4}{2}$ ways of choosing 2 objects, etc. The sum of these are the total number of ways of choosing anything from S, which is the same as the total number of subsets of S. The argument is the same for a set of size n. The total number subsets are

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}.$$

We can sum this by comparing it to the binomial theorem:

$$(x+y)^n = \sum_{k=0}^{k=n} \binom{n}{k} x^k y^{n-k}.$$

The result, 2^n, is obtained immediately by setting $x = 1$ and $y = 1$.

7. Construct a table as in **6** for $S = \{a, b, c, d, e\}$.

8. Make a conjecture for the sum

$$\binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + \cdots + 2^n\binom{n}{n}$$

by studying the pattern for small n. Prove your result by using the binomial theorem.

9. Find a simple expression for this sum using the binomial theorem:

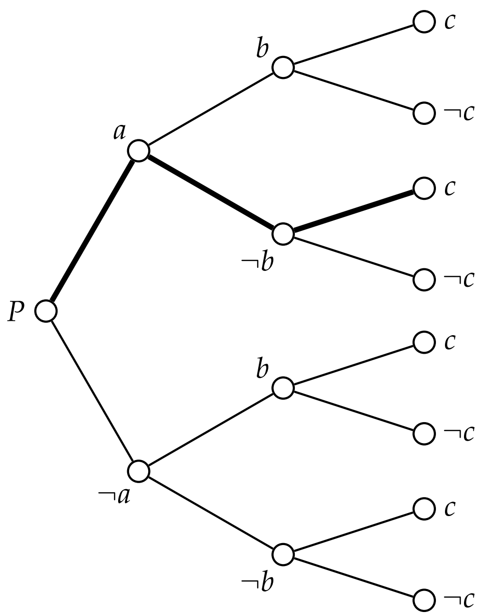
$$\binom{n}{0} + 3\binom{n}{1} + 3^2\binom{n}{2} + \cdots + 3^n\binom{n}{n}.$$

10. Sum the most general case of **8** and **9** using the binomial theorem:

$$\binom{n}{0} + m\binom{n}{1} + m^2\binom{n}{2} + \cdots + m^n\binom{n}{n}.$$

11. Prove the conjecture in **1** by using graphs.

Represent subsets by paths in a graph and then study the pattern that emerges. Let P be the starting node. Every subset of $\{a, b, c\}$ can be represented by a path from P to one of the leaves on the right-hand side. A node labeled a means a is chosen and is in the subset. A node labeled $\neg a$ means a is not chosen and is not in the subset.



The path drawn in bold, $P \to a \to \neg b \to c$, represents the subset $\{a, c\}$. The graph is a tree. In a tree there is only one path from P to any other node. Therefore there are exactly the same number of paths representing subsets as there are leaves, i.e., 2^3. A set S with n elements leads to a tree with 2^n leaves. Therefore S has 2^n subsets.

12. To the right of each leaf in the diagram in **11**, write down the subset corresponding to the path from P to the leaf. For example, the path in bold is $\{a, c\}$.

13. Construct a tree graph representing the subsets of $S = \{a, b, c, d\}$. Label the nodes as in **11** and label the leaves as in **12**.

14. Prove the result in **1** by using binary codes.

An element is either in a particular subset or not. Let `1` mean that an element is in, and `0` mean that it is not. With this idea we can assign a unique code to every subset. As an example, take $S = \{a, b, c\}$. The code `010` means that a is not in the subset, b is in the subset, and c is not in the subset. The code `010` corresponds to the subset $\{b\}$. Here is a table of all subsets with their corresponding binary codes:

subset	binary code
$\{\}$	000
$\{a\}$	100
$\{b\}$	010
$\{c\}$	001
$\{a, b\}$	110
$\{a, c\}$	101
$\{b, c\}$	011
$\{a, b, c\}$	111

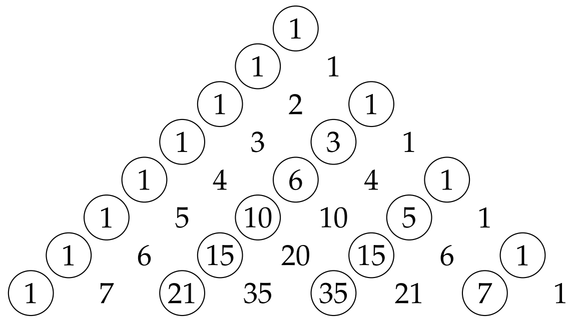
In this table, every subset corresponds to exactly one code, and every code corresponds to exactly one subset. If S has n elements then there are 2^n such codes, and therefore S has 2^n subsets.

15. Make a binary code table as in **14** for the set $\{a, b, c, d\}$.

16. Let $S = \{a, b, c, d\}$. Make a list of all the even-sized subsets of S. Make a list of all the odd-sized subsets of S.

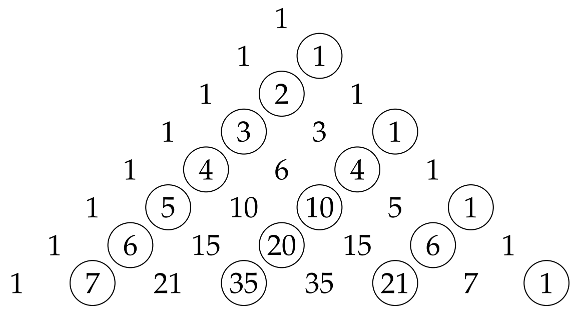
17. Let S be a set of n elements. Let $\alpha(n)$ be the number of even-sized subsets of S. Let $\beta(n)$ be the number of odd-sized subsets of S. Study patterns in $\alpha(n)$ and $\beta(n)$ for various values of n. Form a conjecture and use it to find $\alpha(100)$ and $\beta(100)$.

Let k be the size of the subset. Phrase this problem in terms of modular arithmetic: "k is even" means $k \equiv 0 \mod 2$. Look at the $k \equiv 0$ entries for Pascal's triangle:



Let's take an example of computing $\alpha(n)$. $\alpha(6)$ is the number of even-sized subsets of S where $|S| = 6$. From the circled entries in row $n = 6$ we have $\alpha(6) = 1 + 15 + 15 + 1 = 32$. There are 32 such even-sized subsets.

Examine the odd sizes. The phrase "k is odd" means $k \equiv 1 \mod 2$. Here, the $k \equiv 1$ entries are circled:



An example of computing $\beta(n)$: $\beta(7)$ is the number of odd-sized subsets of S where $|S| = n = 7$. From row $n = 7$ we have $\beta(7) = 7 + 35 + 21 + 1 = 64$. There are 64 odd-sized subsets of a 7-element set S.

It's clear that for any row n, the sum of the $k \equiv 0$ entries and the $k \equiv 1$ entries is the total number of subsets, which is 2^n. So we must have

$$\alpha(n) + \beta(n) = 2^n.$$

Let's make a table to see how $\alpha(n)$ and $\beta(n)$ behave for various values of n.

n	$\alpha(n)$	$\beta(n)$
0	1	0
1	1	1
2	2	$1 + 1 = 2$
3	$1 + 3 = 4$	$3 + 1 = 4$
4	$1 + 6 + 1 = 8$	$4 + 4 = 8$
5	$1 + 10 + 5 = 16$	$5 + 10 + 1 = 16$

From the simple pattern displayed in the table, we can conjecture that, when $n > 0$, the number of even subsets is always equal to the number of odd subsets, and they are both equal to 2^{n-1}.

Keep in mind that $\alpha(n) = \beta(n) = 2^{n-1}$ is a conjecture discovered by looking at a few entries in a table. It isn't a proof.

As pointed out previously, α and β must add up to 2^n. Verify that:

$$\alpha(n) + \beta(n) = 2^{n-1} + 2^{n-1} = 2^n.$$

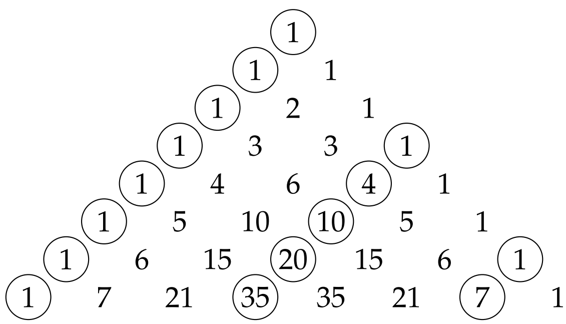
From the conjecture, $\alpha(100) = \beta(100) = 2^{99}$. Although this is only a conjecture, we have no reason to disbelieve it. The pattern in the table comes from the way Pascal's triangle is constructed: by means of a regular law. If we were talking about something far less predictable, such as prime numbers, we wouldn't be so certain.

18. S has 10 elements. How many subsets of odd size does it have?

19. A natural question regarding problem **17** is to ask what would happen if we took mod 3 rather than mod 2.

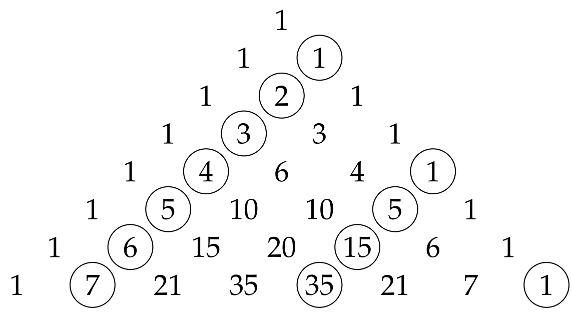
Instead of two possibilities—even and odd sized sets—we now have subsets with sizes congruent to 0, 1 or 2 mod 3. Let S be a set of size n. Define α, β, γ as follows. $\alpha(n)$ is the number of subsets of S having size $k \equiv 0$. $\beta(n)$ is the number of subsets of size $k \equiv 1$. $\gamma(n)$ is the number of subsets of size $k \equiv 2$. Find $\alpha(50)$. Since $k \equiv 0$ mod 3 is the same as saying k is divisible by 3, we can phrase it like so: how many subsets with sizes divisible by 3 does a 50-element set have?

The number of subsets of any size k can be determined from Pascal's triangle. Here are the $k \equiv 0 \mod 3$ entries:



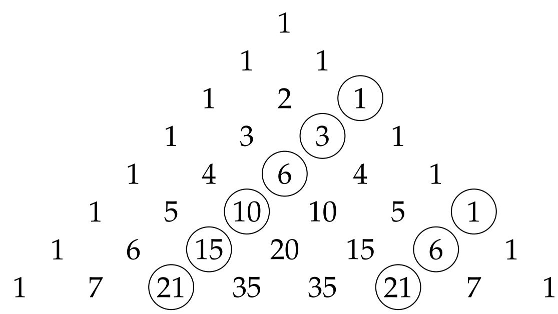
For example, given that $|S| = 5$, what is the total number of subsets of size $k \equiv 0$? Look at the $n = 5$ row and add the circled entries: $1 + 10 = 11$. There are 11 such subsets of S.

The next diagram illustrates the $k \equiv 1$ entries.



If S has 7 elements, how many k-sized subsets does it have such that $k \equiv 1$? Look at row $n = 7$. Add the circled entries: $7 + 35 + 1 = 43$. There are 43 such subsets.

The last case is $k \equiv 2$.



If $S = 6$, how many $k \equiv 2$ subsets does it have? Look at the $n = 6$ row. Add the circled entries, $15 + 6 = 21$. There are 21 such subsets of S.

In problem **17** we made a table to help us discover patterns. In **17** the pattern was very simple. We don't expect the pattern to be so simple this time. Nevertheless let's see if we can guess it from a table of α, β, γ for various values of n.

n	$\alpha(n)$	$\beta(n)$	$\gamma(n)$
0	1	0	0
1	1	1	0
2	1	2	1
3	$1 + 1 = 2$	3	3
4	$1 + 4 = 5$	$4 + 1 = 5$	6
5	$1 + 10 = 11$	$5 + 5 = 10$	$10 + 1 = 11$

There does seem to be a pattern. In every row, two entries are always equal. The other is either bigger or smaller than the equal entries. The unequal entry always seems to be ± 1 relative to the equal entries.

Let's abstract away the numbers and keep only the patterns. Let $\uparrow$ mean $+1$ relative to the others in the row, and let $\downarrow$ mean -1 relative to the others.

n	$\alpha(n)$	$\beta(n)$	$\gamma(n)$
0	$\uparrow$	$=$	$=$
1	$=$	$=$	$\downarrow$
2	$=$	$\uparrow$	$=$
3	$\downarrow$	$=$	$=$
4	$=$	$=$	$\uparrow$
5	$=$	$\downarrow$	$=$

How many of these patterns can there be? There are $\binom{3}{2}$ ways to place the '$=$' symbols. How many ways can we place an arrow? Two ways, because there are two kinds of arrows: $\uparrow$ and $\downarrow$. In all, a row may have

$$\binom{3}{2} \times 2 = 6$$

different possible patterns. Since there are only 6 possible patters for a row, while Pascal's triangle has an unlimited number of rows, we conjecture that the patterns repeat themselves after every 6 rows. If our conjecture is true, the $n = 6$ row should repeat the pattern in row $n = 0$, and row $n = 7$ should repeat the pattern in row $n = 1$.

n	$\alpha(n)$	$\beta(n)$	$\gamma(n)$
6	$1 + 20 + 1 = 22$	$6 + 15 = 21$	$15 + 6 = 21$
7	$1 + 35 + 7 = 43$	$7 + 35 + 1 = 43$	$21 + 21 = 42$

n	$\alpha(n)$	$\beta(n)$	$\gamma(n)$
6	$\uparrow$	$=$	$=$
7	$=$	$=$	$\downarrow$

Which is as expected, but keep in mind that this does not prove that the conjecture is true. Nevertheless, let's go ahead and use the conjecture to find $\alpha(50)$.

Which pattern will row $n = 50$ have? Use Euclidean division:

$$50 = 8 \times 6 + 2.$$

If the conjecture is true, row $n = 50$ will have the same pattern as row $n = 2$. Also, as in **17**, for any n, the sum of all the different sizes of subsets must be 2^n.

Combining these we have,

$$\begin{aligned}\alpha(50) + \beta(50) + \gamma(50) &= \alpha(50) + (\alpha(50) + 1) + \alpha(50) \\ &= 2^{50}.\end{aligned}$$

From which we get the answer:

$$\alpha(50) = \frac{2^{50} - 1}{3} = 375299968947541.$$

A 50-element set S has 375299968947541 subsets that have size divisible by 3.

20. Continue the table in problem **19** until $n = 11$.

21. Find $\beta(50)$ and $\gamma(50)$.

22. Find $\alpha(130)$, $\beta(130)$ and $\gamma(130)$.

2 Row patterns

23. Draw Pascal's triangle. Make a conjecture for the sum of the squares of the binomial numbers in each row:

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2.$$

24. Prove the conjecture you got in **23** using a combinatorial argument.

The conjecture you should have obtained is that the sum of the squares of the binomial numbers in row n is a special binomial number called the *central binomial number*. Here are the first few central binomial numbers:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

The central binomial number is the number of ways of choosing n objects from a total of $2n$ objects. We can divide the $2n$ objects into two groups of n objects,

○ ○ ○ ○ ○ ○ ○ ○ ○ ○

and count the ways of choosing n of them according to a different plan. We take $n = 5$ just to make the illustrations simple, but the argument is exactly the same for any n. The black dots represent the choices we make. The white dots represent the objects we are choosing from. Choose n from the first group and none from the second:

○ ○ ○ ○ ○ ○ ○ ○ ○ ○

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The number of ways of doing this is

$$\binom{n}{n} \times \binom{n}{0}.$$

Now choose $n - 1$ objects from the first group and 1 object from the second group:

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The number of ways of doing this is

$$\binom{n}{n-1} \times \binom{n}{1}.$$

We can continue this until we choose 0 objects from the first group and n from the second group:

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The number of ways of doing this is

$$\binom{n}{0} \times \binom{n}{n}.$$

The sum of all these possibilities is the number of ways of choosing n objects from $2n$ objects.

$$\binom{n}{n}\binom{n}{0} + \binom{n}{n-1}\binom{n}{1} + \cdots + \binom{n}{0}\binom{n}{n} = \binom{2n}{n}.$$

Finally, by exploiting the symmetry in the rows of Pascal's triangle as expressed by the fundamental identity

$$\binom{n}{k} = \binom{n}{n-k}$$

we have

$$\binom{n}{n}\binom{n}{n} + \binom{n}{n-1}\binom{n}{n-1} + \cdots + \binom{n}{0}\binom{n}{0} = \binom{2n}{n}.$$

Which is the conjecture that that you should have obtained in **23** by studying patterns:

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}. \tag{1}$$

We have interpreted the right hand side and the left hand side of this conjecture in terms of combinatorics, and we showed that both sides are just different ways of counting the same thing. This is a typical strategy in combinatorial proofs.

25. Verify that the left-hand side and right-hand side are equal:

$$\binom{5}{5}\binom{5}{0} + \binom{5}{4}\binom{5}{1} + \cdots + \binom{5}{0}\binom{5}{5} = \binom{10}{5}.$$

26. In **24** we took a collection of $2n$ objects and separated them into two groups of n objects. What would happen if we separated them into two *unequal* groups?

We will work through a small numerical example to show that this idea leads to other binomial relationships. Take $n = 5$ and $2n = 10$, but divide the objects into a group of four and a group of six. Start by making all choices from the second group. In other words, choose none from the first group and five from the second group:

○○○○ ○○○○○○

●●●●●

There are

$$\binom{4}{0} \times \binom{6}{5}$$

ways to do this. Next, choose one object from the first group, and four objects from the second group:

○ ○ ○ ○   ○ ○ ○ ○ ○ ○

●   ● ● ● ●

There are

$$\binom{4}{1} \times \binom{6}{4}$$

ways to do this. Continue this argument until we reach the last arrangement, which is when four objects are chosen from the left group and one from the right. We have to stop here because we only have four objects in the left group. One choice is left over, so it stays on the right:

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The number of ways to do this is

$$\binom{4}{4} \times \binom{6}{1}.$$

It's clear that all this is just another method for organizing how to count the ways of choosing 5 objects from 10 objects. Therefore we have

$$\binom{4}{0}\binom{6}{5} + \binom{4}{1}\binom{6}{4} + \cdots + \binom{4}{4}\binom{6}{1} = \binom{10}{5}.$$

Let's verify this. The left-hand side is

$$1 \times 6 + 4 \times 15 + 6 \times 20 + 4 \times 15 + 1 \times 5 = 252.$$

The right hand side, $\binom{10}{5}$, is also 252.

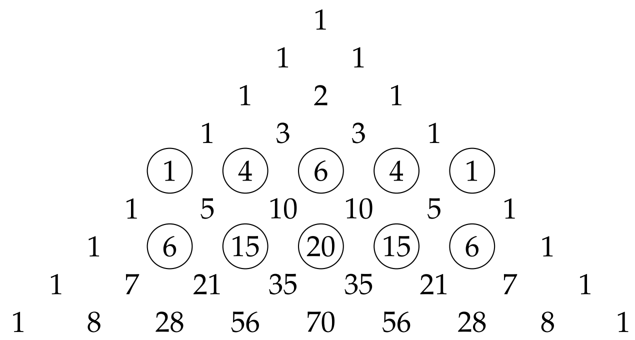
27. Repeat problem **26** but with $n = 6$. In this case we are choosing $n = 6$ objects from $2n = 12$ objects, separated unequally into a group of five and a group of seven.

28. Give a combinatorial proof for the identity

$$\binom{n-1}{0}\binom{n+1}{1} + \binom{n-1}{1}\binom{n+1}{2} + \cdots$$
$$\cdots + \binom{n-1}{n-1}\binom{n+1}{n} = \binom{2n}{n}$$

based on the methods of problems **24** and **26**. You may need to use the Pascal symmetry relationship $\binom{n}{k} = \binom{n}{n-k}$.

29. Consider separated rows in Pascal's triangle.

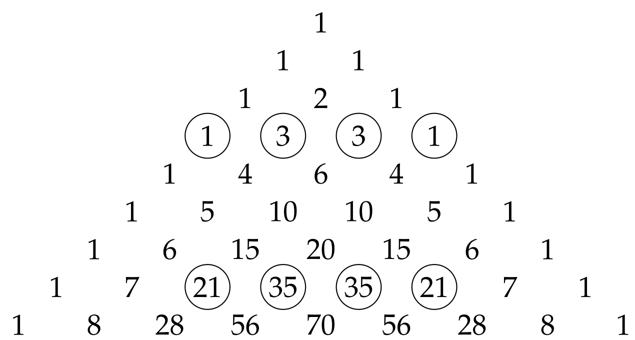


Here $n = 5$. We circled the elements in rows $n-1$ and $n+1$. If we multiply corresponding circled elements and take the sum, we get

$$1 \times 6 + 4 \times 15 + 6 \times 20 + 4 \times 15 + 1 \times 6.$$

Draw Pascal's triangle and study this sum for various values of n. Make a conjecture.

30. As in problem **29**, except the rows are taken farther apart: $n-2$ and $n+2$ with $n = 5$:



Study the same kind of sum as in problem **29** for various values of n. Make a conjecture.

31. Combine the results of problems **29** and **30** into one conjecture for rows $n-r$, $n+r$.

32. For problem **31** you should have obtained

$$\binom{n-r}{0}\binom{n+r}{r}+\binom{n-r}{1}\binom{n+r}{r+1}+\cdots$$
$$\cdots+\binom{n-r}{n-r}\binom{n+r}{n}=\binom{2n}{n}.$$

This can be written more compactly as

$$\sum_{j=0}^{n-r}\binom{n-r}{j}\binom{n+r}{j+r}=\binom{2n}{n}. \tag{2}$$

This spectacular identity reduces to (1) when r is set to 0. Prove this identity using the methods of **26**, **27** and **28**.

33. Prove the following identity by using combinatorial arguments:

$$\sum_{j=0}^{k}\binom{m}{j}\binom{n}{k-j}=\binom{m+n}{k}. \tag{3}$$

This famous relationship is called the Vandermonde identity. It is similar in spirit to (2) and can be proved by the same kind of combinatorial arguments used to prove both (1) and (2).

In problem **26** we considered selecting n objects from $2n$ objects. We saw what happens when the $2n$ objects are separated into two unequal groups. We developed this idea further until problem **31**, where we separated the $2n$ objects into two groups this way:

$$2n=(n+r)+(n-r).$$

But what would happen if we removed the constraint that the objects have to add up to $2n$? What if we simply had two groups of objects and we select k from them? This leads to the Vandermonde identity.

Suppose we have a total of $m+n$ objects. We can draw these objects conceptually separated into two groups:

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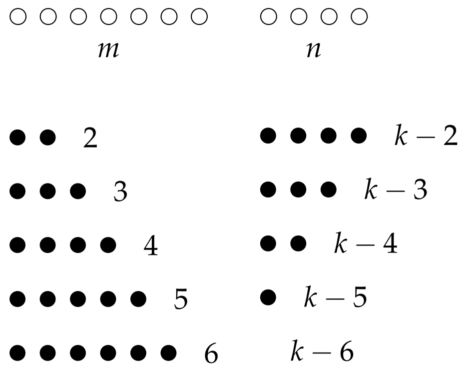
m n

For the sake of illustration we take $m=7$, $n=4$ and we will chose $k=6$ objects from these $m+n=11$ objects. Start by choosing two objects from the first group and the rest from the second group. That's the minimum number of objects that we can choose from the first group. Then we proceed to draw all the other possibilites and arrange the results in a table.



This covers all the ways to choose $k = 6$ objects from the two groups. Therefore the sum of all the ways to make the choices shown in the rows of the above table is simply the number of ways to choose k from $m + n$:

$$\binom{m}{2}\binom{n}{k-2} + \binom{m}{3}\binom{n}{k-3} + \cdots$$
$$\cdots + \binom{m}{6}\binom{n}{k-6} = \binom{m+n}{k}.$$

The above argument is perfectly general. It holds for any m, n and k. Therefore we can write our result as:

$$\sum_{j=0}^{k} \binom{m}{j}\binom{m}{k-j} = \binom{m+n}{k}.$$

Notice that we started the summation at $j = 0$, which is very convenient, but we need to justify this. Starting at $j = 0$ gives extra leading terms, but they are all zero, because these terms correspond to choosing $k > n$ objects from the group of n. Such as, for example, $k = 6$ objects from the group of $n = 4$ objects. There are zero ways to do that.

34. Show that by setting $m = n$ and $k = n$, the Vandermonde identity (3) becomes (1).

35. Draw Pascal's triangle. Choose various values of n and study the following sum:

$$\binom{n}{0}\binom{n}{1} + \binom{n}{1}\binom{n}{2} + \cdots + \binom{n}{n-1}\binom{n}{n}.$$

Make a conjecture for the value of this sum.

36. Draw Pascal's triangle. Study the following sum for various values of n:

$$\binom{n}{0}\binom{n}{2} + \binom{n}{1}\binom{n}{3} + \cdots + \binom{n}{n-2}\binom{n}{n}.$$

Make a conjecture.

37. Draw Pascal's triangle. Study this sum for various values of n:

$$\binom{n}{0}\binom{n}{3} + \binom{n}{1}\binom{n}{4} + \cdots + \binom{n}{n-3}\binom{n}{n}.$$

Make a conjecture.

38. Using two variables, n and m, find a way to combine the conjectures of problems **35**, **36** and **37** into one conjecture that covers all of them.

39. Use the factorial definition of binomial numbers

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{4}$$

to prove the identity

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}. \tag{5}$$

40. Prove (5) by a combinatorial argument.

We have n people. How many ways can we select a team of k players and a team captain? Let's do it by first selecting a team, and then a captain from among the team members. There are $\binom{n}{k}$ ways to choose a team of k players from n people. From these team members, there are k ways to choose a captain. So, the number of ways of choosing a team first, followed by a team captain is

$$\binom{n}{k} \times k.$$

Now let's choose the captain first, followed by the team. There are n ways to choose a team captain from n people. Since one choice and one team member is accounted for, there are $k-1$ players to be chosen from $n-1$ people. The number of ways of choosing a captain first, followed by the rest of the team is

$$n \times \binom{n-1}{k-1}.$$

We are counting the same thing in two different ways, so both ways must be equal:

$$\binom{n}{k} \times k = n \times \binom{n-1}{k-1}.$$

41. Draw Pascal's triangle and use it to study the pattern for this sum by looking at various values of n:

$$0\binom{n}{0} + 1\binom{n}{1} + \cdots + k\binom{n}{k} + \cdots + n\binom{n}{n}.$$

Make a conjecture for the value of this sum.

42. Give an algebraic proof of the conjecture in **41**.

The expression in **41** can be neatly written as

$$0\binom{n}{0} + 1\binom{n}{1} + \cdots + n\binom{n}{n} = \sum_{k=0}^{n} k\binom{n}{k}.$$

Since the first term is 0, we can start the summation at $k = 1$:

$$\sum_{k=1}^{n} k\binom{n}{k}.$$

That is what we want to evaluate. Applying identity (5) from problem **39** eliminates k in each term:

$$\sum_{k=1}^{n} k\binom{n}{k} = \sum_{k=1}^{n} k\frac{n}{k}\binom{n-1}{k-1}$$

$$= n\sum_{k=1}^{n} \binom{n-1}{k-1}.$$

Change the variable of summation to j by using $j = k - 1$:

$$n\sum_{k=1}^{n} \binom{n-1}{k-1} = n\sum_{j=0}^{n-1} \binom{n-1}{j}.$$

The sum

$$\sum_{j=0}^{n-1} \binom{n-1}{j}$$

is the sum of all binomial numbers in row $n - 1$, which is 2^{n-1}, because the sum of the entries in row m of Pascal's triangle is 2^m. Therefore we have:

$$n\sum_{j=0}^{n-1} \binom{n-1}{j} = n2^{n-1}.$$

43. Give a combinatorial proof for the conjecture you obtained in problem **41**.

Combinatorial arguments often lead to very elegant proofs—you will see another example of that here.

Let's say we have n players. How many ways can we choose a team (of any size up to n) and a team captain from these players? Suppose we choose the captain first. There are n ways to do that. Since we already have the captain, we have to choose a team from the remaining $n-1$ players. The team can be of any size, so this is exactly like choosing a subset from a set of $n-1$ elements. There are 2^{n-1} ways to do that. In all there are

$$n \times 2^{n-1}$$

ways to choose a captain first followed by choosing the rest of the team.

Now choose the team first and then a captain from among the team members. There are $\binom{n}{k}$ ways to choose a team of size k. And there are k ways to choose a captain from the members of that team. If we add all the possibilities for every size of team together, we have:

$$0\binom{n}{0} + 1\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n}.$$

But this is just another way of counting teams and captains, so it must be equal to $n2^{n-1}$,

$$0\binom{n}{0} + 1\binom{n}{1} + \cdots + n\binom{n}{n} = n2^{n-1}$$

or, using summation notation:

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}. \tag{6}$$

44. In problem **40** we considered teams with captains. Specifically, one captain for each team. What happens if we want two captain per team? Develop a combinatorial argument similar to the one in **40** but with two captains.

It sounds strange to say that a team has more than one captain. This is an example were a word-problem or a practical analogy of a mathematical idea can lead you in the wrong direction. In our common experience, the captain of a team is unique. But there is no such connotation in the argument of **40**. Mathematically, the "captain" is an element of a subset that is designated as special in some way. It's possible to have more than one special element of course. If we take the analogy of "captains" and "teams" too literally, we might never get the idea to examine what happens when we have more than one captain. Here we see that mathematical ideas are sometimes imperfectly captured by common analogies.

Suppose we have a total of n players to choose from. Because we have said that each team must have two captains, the smallest possible team will have two people. There are $\binom{n}{2}$ ways to choose this team, and then there are $\binom{2}{2}$ ways to designate the two captains from among the team members (i.e., there is only one way.) Therefore the number of ways to choose a team of size 2 is:

$$\binom{2}{2} \times \binom{n}{2}.$$

Next, how many ways are there to make teams of size 3, with two captains? There are $\binom{n}{3}$ ways to choose the teams, and then $\binom{3}{2}$ ways to designate the captains from among the team members. Therefore the number of ways to do this is:

$$\binom{3}{2} \times \binom{n}{3}.$$

We continue this way until we get to the largest possible team, which is a team of n players. The number of ways to choose this team is $\binom{n}{n}$, while the number of ways to choose the captains is $\binom{n}{2}$. In all we have these possibilities:

$$\binom{n}{2} \times \binom{n}{n}.$$

The sum of all these terms is the number of ways to form any size of team with two captains. This can be counted a different way by first choosing the captains from among the n players: there are $\binom{n}{2}$ ways to do that. Then choose the rest of the team from among the $n-2$ players left: there are 2^{n-2} ways to do that. Putting all this together, we have

$$\binom{2}{2}\binom{n}{2} + \binom{3}{2}\binom{n}{3} + \cdots + \binom{n}{2}\binom{n}{n} = \binom{n}{2}2^{n-2}$$

or,

$$\sum_{j=0}^{n} \binom{j}{2}\binom{n}{j} = \binom{n}{2}2^{n-2}.$$

We can start the sum at $j = 0$ because the extra leading terms are zero anyway.

45. Develop the same argument as in **44**, but with three captains. What identity do you get?

46. Generalize the arguments of problems **44** and **45** to the case of m captains. What identity do you obtain?

Since the number of captains, m, is a fixed number, it is convenient to write the total number of players, n, in terms of m like so: $n = m + r$. The smallest possible team has m players and the number of ways to chooose this team along with its captains is

$$\binom{m}{m}\binom{m+r}{m}.$$

The largest possible team has $m + r$ players. The number of ways to choose this team, along with its captains is:

$$\binom{m+r}{m}\binom{m+r}{m+r}.$$

The total these terms and all the ones in the middle can be counted a different way. There are $\binom{m+r}{m}$ ways to choose the captains first, and then 2^r ways to choose the rest of the team. So we have

$$\binom{m}{m}\binom{m+r}{m}+\binom{m+1}{m}\binom{m+r}{m+1}+\cdots$$
$$\cdots+\binom{m+r}{m}\binom{m+r}{m+r}=\binom{m+r}{m}2^r.$$

With summation notation this can be written nicely:

$$\sum_{j=0}^{r}\binom{m+j}{m}\binom{m+r}{m+j}=\binom{m+r}{m}2^r. \tag{7}$$

Again, we have chosen to start the sum at $j = 0$, since those extra leading terms are zero anyway.

47. Show that when $m = 0$, identity (7) reduces to the well-known relationship that the sum of the elements in row r of Pascal's triangle is 2^r.

48. Show that when $m = 1$, identity (7) becomes (6).

49. Let $i \geq k \geq j$. Prove the identity

$$\binom{i}{k}\binom{k}{j}=\binom{i}{j}\binom{i-j}{k-j}. \tag{8}$$

by a combinatorial argument.

The first step is to understand what is going on in (8). The left-hand side says that we are choosing k objects from i objects, and then we choose j objects from those k objects. We can phrase it like this: from a total of i objects, choose a collection of size k, and then from this collection, choose a subcollection of size j. This explains why there is the condition $i \geq k \geq j$.

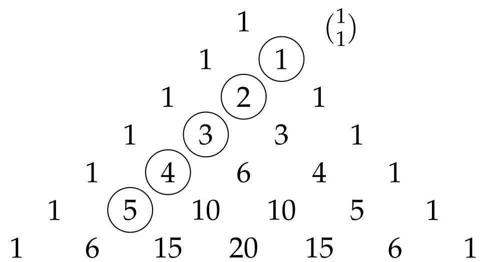
We are choosing a collection of objects first, and then from these we choose a subcollection. What if we choose the subcollection first? There are $\binom{i}{j}$ ways to do that. Since we have already chosen j objects, there are $i - j$ objects left to choose from. Since the subcollection is part of the collection, j elements have been accounted for, and there are $k - j$ elements left to choose for the collection. Therefore we have $\binom{i-j}{k-j}$ ways to fill in the rest of the collection. The total number of ways of choosing the subcollection first, followed by the collection, is the product of the two binomial expression. Therefore we get (8).

50. Give an algebraic proof of (8) by using the factorial definition of binomial numbers (4).

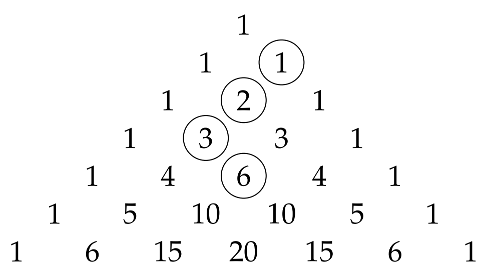
51. Using (8), give an algebraic proof of (7).

3 Diagonal patterns

52. Consider the diagonal in Pascal's triangle starting at $\binom{1}{1}$ and going to the lower left. Make a conjecture about the sum of the elements in this diagonal.



Start by examining a small section of the diagonal.



We see that the sum of the circled elements in the diagonal is 6, which happens to be an element in the next row. Write this finding down:

$$\binom{1}{1} + \binom{2}{1} + \binom{3}{1} = \binom{4}{2}.$$

Now take a longer section of the diagonal.

$$\begin{array}{ccccccccccccc}
&&&&&&1&&&&&&\\
&&&&&1&&\textcircled{1}&&&&&\\
&&&&1&&\textcircled{2}&&1&&&&\\
&&&1&&\textcircled{3}&&3&&1&&&\\
&&1&&\textcircled{4}&&6&&4&&1&&\\
&1&&\textcircled{5}&&10&&10&&5&&1&\\
1&&6&&\textcircled{15}&&20&&15&&6&&1
\end{array}$$

As before, the sum of the elements in the diagonal gives an element of the next row:

$$\binom{1}{1} + \binom{2}{1} + \binom{3}{1} + \binom{4}{1} + \binom{5}{1} = \binom{6}{2}.$$

We can put these patterns together to get a general conjecture for any length r of the diagonal:

$$\binom{1}{1} + \binom{2}{1} + \cdots + \binom{r}{1} = \binom{r+1}{2}.$$

53. As in **52** but starting at $\binom{2}{2}$. Make a conjecture about the sum of the elements in this diagonal:

$$\begin{array}{cccccccccccccc}
&&&&&&1&&&&&&&\\
&&&&&1&&1&&\binom{2}{2}&&&&\\
&&&&1&&2&&\textcircled{1}&&&&&\\
&&&1&&3&&\textcircled{3}&&1&&&&\\
&&1&&4&&\textcircled{6}&&4&&1&&&\\
&1&&5&&\textcircled{10}&&10&&5&&1&&\\
1&&6&&\textcircled{15}&&20&&15&&6&&1&
\end{array}$$

54. As in **52** but starting at $\binom{3}{3}$. Make a conjecture about the sum of the elements in this diagonal:

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & \binom{3}{3} & & \\
 & & & 1 & & 3 & & 3 & & \textcircled{1} & & & \\
 & & 1 & & 4 & & 6 & & \textcircled{4} & & 1 & & \\
 & 1 & & 5 & & 10 & & \textcircled{10} & & 5 & & 1 & \\
1 & & 6 & & 15 & & \textcircled{20} & & 15 & & 6 & & 1
\end{array}$$

55. Study the patterns in **52**, **53** and **54**. Make a conjecture for the sum of any diagonal beginning at $\binom{m}{m}$.

Collecting the previous conjectures

$$\binom{1}{1} + \binom{2}{1} + \cdots + \binom{r}{1} = \binom{r+1}{2}$$

$$\binom{2}{2} + \binom{3}{2} + \cdots + \binom{r}{2} = \binom{r+1}{3}$$

$$\binom{3}{3} + \binom{4}{3} + \cdots + \binom{r}{3} = \binom{r+1}{4}$$

and studying the pattern, we see that they can be combined into one conjecture using an offset m that tells us where to begin:

$$\binom{m}{m} + \binom{m+1}{m} + \cdots + \binom{m+n}{m} = \binom{m+n+1}{m+1}. \tag{9}$$

This is an important formula which forms the basis for solving other types of problems. It is well worth remembering, but if you ever forget it, you can always re-discover it by studying patterns in Pascal's triangle.

56. Give a combinatorial argument for the identity

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}. \tag{10}$$

This is the most important of all identities involving Pascal's triangle. In fact, this formula is a mathematical expression for the way we construct Pascal's triangle by adding entries in the previous row to get an entry in the current row. For example, $n = 4$,

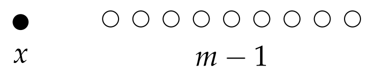
$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & \textcircled{4} & & \textcircled{6} & & 4 & & 1 & \\
1 & & 5 & & \textcircled{10} & & 10 & & 5 & & 1
\end{array}
$$

and we have

$$10 = \binom{5}{2} = \binom{4}{1} + \binom{4}{2} = 4 + 6.$$

We could say that identity (10) follows from the definition of Pascal's triangle. But we need to go a bit further and prove that the elements in Pascal's triangle are not just numbers obtained by applying a peculiar rule—they have a special combinatorial interpretation.

We do this by giving (10) an interesting combinatorial meaning. Take m objects and designate one object as special. Call the special object x. Our m objects can be conceptually separated into object x and the other $m-1$ objects:



How many ways can we select k objects from m objects such that x is among them? We can be sure that x is among them by choosing x first. We have now used one of our choices: there are $k-1$ choices left, and these choices must be from the $m-1$ other objects. Therefore there are

$$\binom{m-1}{k-1}$$

ways to choose k objects from m objects such that the special object x is always among them.

Next, how many ways can we choose k objects from m objects such that x is *not* among them? This is simple. We make all our k choices from the group of $m-1$ objects to the right of x in the figure. There are

$$\binom{m-1}{k}$$

ways to make this choice. Clearly, if we add all the ways of choosing *with* x and all the ways of choosing *without* x, we get all the ways of choosing k objects from from m objects:

$$\binom{m}{k} = \binom{m-1}{k-1} + \binom{m-1}{k}.$$

Finally, substitute $m = n + 1$ to get (10).

57. Give an algebraic proof of (9).

A quick look at (9) gives the impression that it may be hard to prove algebraically. But it isn't. To prove it we need to apply (10) repeatedly. The process is easier to visualize if we begin by rearranging (9) as

$$\binom{m+n+1}{m+1} = \binom{m+n}{m} + \binom{m+n-1}{m} + \cdots + \binom{m}{m}.$$

Apply (10) to the left-hand side:

$$\binom{m+n+1}{m+1} = \binom{m+n}{m} + \binom{m+n}{m+1}.$$

The first term on the right-hand side of this is exactly what we want. But the second term is not. Apply (10) to the second term:

$$\binom{m+n+1}{m+1} = \binom{m+n}{m} + \binom{m+n-1}{m} + \binom{m+n-1}{m+1}.$$

The first two terms on the right-hand side are what we want. Keep applying (10) to the last terms on the right-hand side, again and again, until finally:

$$\binom{m+n+1}{m+1} = \binom{m+n}{m} + \cdots + \binom{m+1}{m} + \binom{m+1}{m+1}.$$

But $\binom{m+1}{m+1}$ is the same as $\binom{m}{m}$. We have,

$$\binom{m+n+1}{m+1} = \binom{m+n}{m} + \cdots + \binom{m+1}{m} + \binom{m}{m}.$$

And this is just another way of writing (9).

58. Give a combinatorial proof of the conjecture discovered in problem **52**.

In **52** we found that

$$\binom{1}{1} + \binom{2}{1} + \cdots + \binom{n}{1} = \binom{n+1}{2}.$$

This is a special case of (9) with m set to 1. We will interpret the right-hand side and the left-hand side in terms of combinatorics and show that both sides count the same thing.

Call a subset of size k a *k-subset* for short. The right-hand side $\binom{n+1}{2}$ is the number of different 2-subsets of a set S that has $n+1$ elements. The $n+1$ elements of S can be anything. Rather than colored balls or symbols, use consecutive integers, because they have nice properties which can be used to make a proof easier:

$$S = \{1, 2, \ldots n+1\}.$$

S has a greatest element, $n+1$. Any subset of S has a greatest element. This is one of those nice properties of integers that we mentioned. We can organize the counting of 2-subsets in terms of greatest elements. Call the greatest element g.

How many 2-subsets have g of 1? It's impossible to form a 2-subset with $g = 1$, because there is no choice in S for the other element. What would you choose from S that is less than 1? So the number of 2-subsets that have $g = 1$ is 0.

How many 2-subsets have $g = 2$? There is only one choice for the other element: 1. Therefore there is only one 2-subset with $g = 2$.

How many 2-subsets have $g = 3$? We have chosen the element 3, now we must make one choice from the two elements in S that are smaller than 3. There are $\binom{2}{1}$ way to do this.

It's easier to see the flow of this argument if we put these results into a table.

g	2-subsets	#
1	none	0
2	$\{1,2\}$	$\binom{1}{1}$
3	$\{1,3\}$, $\{2,3\}$	$\binom{2}{1}$
4	$\{1,4\}$, $\{2,4\}$, $\{3,4\}$	$\binom{3}{1}$
n	$\{1,n\},\ldots \{n-1,n\}$	$\binom{n-1}{1}$
$n+1$	$\{1,n+1\},\ldots \{n,n+1\}$	$\binom{n}{1}$

Clearly, this is just organizing all the 2-subsets by what g they contain. The sum of all the different ways of obtaining 2-subsets of S for every g is just the total number of 2-subsets of S. Therefore

$$\binom{1}{1} + \binom{2}{1} + \cdots + \binom{n-1}{1} + \binom{n}{1} = \binom{n+1}{2}.$$

59. In problem **53** you should have obtained the conjecture

$$\binom{2}{2} + \binom{3}{2} + \cdots + \binom{n}{2} = \binom{n+1}{3}.$$

Develop a combinatorial argument to prove this. Use the argument in **58** as a model and make an analogous table.

60. In problem **54** you should have got the conjecture

$$\binom{3}{3}+\binom{4}{3}+\cdots+\binom{n}{3}=\binom{n+1}{4}.$$

Develop a combinatorial proof of this based on the one given in **58**. Also, make the same kind of table as in **58**.

61. Use the work of problems **58**, **59**, and **60** to develop a combinatorial proof for the general case of (9).

62. Consider the diagonal in Pascal's triangle starting at $\binom{1}{0}$ and going to the lower right. Make a conjecture about the sum of the elements in this diagonal.

$\binom{1}{0}$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

63. As in problem **62**, make a conjecture about the sum of the elements in this diagonal:

$\binom{2}{0}$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

64. As you did in problem **62**, make a conjecture about the sum of the elements in the diagonal beginning at $\binom{3}{0}$:

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & \binom{3}{0} & & & 1 & & 2 & & 1 & & & & \\
 & & & \textcircled{1} & & 3 & & 3 & & 1 & & & \\
 & & 1 & & \textcircled{4} & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & \textcircled{10} & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & \textcircled{20} & & 15 & & 6 & & 1
\end{array}
$$

65. Study the patterns you obtained in problems **62**, **63** and **64** and write down a conjecture for the sum of any diagonal beginning at $\binom{m}{0}$ and going down to the lower right.

From the above-mentioned problems you should have obtained the conjectures

$$\binom{1}{0} + \binom{2}{1} + \cdots + \binom{n+1}{n} = \binom{n+2}{2}$$

$$\binom{2}{0} + \binom{3}{1} + \cdots + \binom{n+2}{n} = \binom{n+3}{3}$$

$$\binom{3}{0} + \binom{4}{1} + \cdots + \binom{n+3}{n} = \binom{n+4}{4}.$$

They can be combined into one conjecture by introducing an offset m telling us where to begin:

$$\binom{m}{0} + \binom{m+1}{1} + \cdots + \binom{m+n}{n} = \binom{m+n+1}{n}. \tag{11}$$

It is interesting to compare (11) with (9):

$$\binom{m}{m} + \binom{m+1}{m} + \cdots + \binom{m+n}{m} = \binom{m+n+1}{m+1}.$$

Compare them term-by-term. The pattern is remarkable. If we add the lower parts of the binomial numbers in each corresponding term of (11) and (9), we get the upper part. This gives us a way to check if we remembered these identities correctly.

66. Use the fundamental binomial symmetry relation

$$\binom{n}{k} = \binom{n}{n-k}$$

to transform (9) into (11).

67. Give an algebraic proof of (11) using the method of problem **57**.

4 Applications

68. Sum this series:

$$1 + 1 + 1 + \cdots + 1$$

where there are $n + 1$ ones.

Of course this is trivial: the answer is simply $n + 1$. But we can use this simple problem to introduce a technique for evaluating many other more complex and interesting sums.

If we can write every term of a series as binomial numbers or some combination of binomial numbers, we might be able to use one of the many binomial identities to sum the series. We will call this idea the *binomial method* for summing series.

The terms of the series $1 + 1 + \cdots + 1$ are entries in Pascal's triangle:

$$\begin{array}{ccccccccccccc}
 & & & & & & \textcircled{1} & & & & & & \\
 & & & & & \textcircled{1} & & 1 & & & & & \\
 & & & & \textcircled{1} & & 2 & & 1 & & & & \\
 & & & \textcircled{1} & & 3 & & 3 & & 1 & & & \\
 & & \textcircled{1} & & 4 & & 6 & & 4 & & 1 & & \\
 & \textcircled{1} & & 5 & & 10 & & 10 & & 5 & & 1 & \\
\textcircled{1} & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

It's easy to write $1 + 1 + \cdots + 1$ in terms of binomial numbers just by looking at Pascal's triangle:

$$\binom{0}{0} + \binom{1}{0} + \binom{2}{0} + \cdots + \binom{n}{0}.$$

Now we can apply one of the diagonal identities, (9):

$$\binom{m}{m} + \binom{m+1}{m} + \cdots + \binom{m+n}{m} = \binom{m+n+1}{m+1}.$$

If we set $m = 0$, the left-hand side becomes the series we are trying to evaluate, while the right hand side is the sum that we are looking for:

$$\begin{aligned}
\binom{0}{0} + \binom{1}{0} + \cdots + \binom{n}{0} &= \binom{0+n+1}{0+1} \\
&= \binom{n+1}{1} = n + 1.
\end{aligned}$$

This may seem like a long way to get a simple answer, but it is a powerful technique that can be used on harder problems.

Just to be complete, we should see how to work this problem out using the *other* diagonal, since it may just as well represent the terms of the original series $1 + 1 + \dots$

$$
\begin{array}{ccccccccccccc}
 & & & & & & \textcircled{1} & & & & & & \\
 & & & & & 1 & & \textcircled{1} & & & & & \\
 & & & & 1 & & 2 & & \textcircled{1} & & & & \\
 & & & 1 & & 3 & & 3 & & \textcircled{1} & & & \\
 & & 1 & & 4 & & 6 & & 4 & & \textcircled{1} & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & \textcircled{1} & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & \textcircled{1}
\end{array}
$$

Using this diagonal, the series $1 + 1 + \cdots + 1$ can be written as:

$$\binom{0}{0} + \binom{1}{1} + \binom{2}{2} + \cdots + \binom{n}{n}.$$

Now is the moment to apply the *other* diagonal identity that we discovered, (11):

$$\binom{m}{0} + \binom{m+1}{1} + \cdots + \binom{m+n}{n} = \binom{m+n+1}{n}.$$

Set $m = 0$ and we obtain

$$\begin{aligned}
\binom{0}{0} + \binom{1}{1} + \cdots + \binom{n}{n} &= \binom{0+n+1}{n} \\
&= \binom{n+1}{n} = n+1.
\end{aligned}$$

It's remarkable how all of this works out to give $n + 1$.

69. Express r in terms of binomial coefficients.

It's important have a technique for expressing powers of r in terms of binomial numbers. We have already seen how to express r^0. Any binomial number on the leftmost or rightmost diagonals of Pascal's triangle will do because r^0 is just 1. Now, what about r^1?

The problem of expressing any power of r in terms of binomial numbers can be solved with a technique that we call *undetermined coefficients*. Assume that any such r^k can be written as a linear combination of binomial coefficients

$$\binom{r}{k}, \binom{r}{k-1}, \binom{r}{k-2}, \dots \binom{r}{1}.$$

Why do we assume this? Because the highest power of r that appears in $\binom{r}{k}$ is k. For example:

$$\binom{r}{2} = \frac{r(r-1)}{2!} = \frac{r^2 - r}{2}$$

and

$$\binom{r}{3} = \frac{r(r-1)(r-2)}{3!} = \frac{r^3 - 3r^2 + 2r}{6}.$$

We do not need to consider any binomial numbers with degree larger than k. Therefore a term like r^k can be written as some kind of combination of kth degree binomial numbers and lower:

$$r^k = A\binom{r}{k} + B\binom{r}{k-1} + C\binom{r}{k-2} + \cdots .$$

The coefficients $A, B, C, \ldots$ have to be determined. It is straightforward to do so.

For this problem, $k = 1$ and we have

$$r^1 = A\binom{r}{1} = A\frac{r}{1!}$$

so obviously $A = 1$ and the answer is simply

$$r = \binom{r}{1}.$$

70. Express r^2 in terms of binomial numbers.

Write r^2 as a linear combination of binomial numbers with undetermined coefficients:

$$\begin{aligned} r^2 &= A\binom{r}{2} + B\binom{r}{1} \\ &= \frac{A}{2}r^2 - \frac{A}{2}r + Br \end{aligned}$$

from which we get equations for A and B:

$$\frac{A}{2} = 1, \quad -\frac{A}{2} + B = 0.$$

We have $A = 2$ and $B = 1$. Therefore

$$r^2 = 2\binom{r}{2} + \binom{r}{1}.$$

71. Express r^3 in terms of binomial numbers. Use the method of undetermined coefficients.

72. Express r^4 in terms of binomial numbers. Use the method of undetermined coefficients.

73. Sum this series using the binomial method:

$$1 + 2 + \cdots + k.$$

Each term in the series is of the form r. In problem **69** we saw how to express r in terms of binomials:

$$r = \binom{r}{1}.$$

With this fact, the series can be expressed as

$$\binom{1}{1} + \binom{2}{1} + \cdots + \binom{k}{1}.$$

This is immediately summed by setting $m = 1$ and $m + n = k$ in diagonal identity (9):

$$\binom{1}{1} + \binom{2}{1} + \cdots + \binom{k}{1} = \binom{k+1}{2},$$

which gives the famous formula for the sum of consecutive integers:

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}. \tag{12}$$

74. Sum the squares of the first k integers

$$1^2 + 2^2 + \cdots + k^2$$

using the binomial method and the result of problem **70**.

In problem **70** we showed that r^2 can be represented as binomials:

$$r^2 = 2\binom{r}{2} + \binom{r}{1}.$$

With this, every term of the sum can be transformed into binomials:

$$2\binom{1}{2} + \binom{1}{1} + 2\binom{2}{2} + \binom{2}{1} + \cdots + 2\binom{k}{2} + \binom{k}{1}.$$

Re-arranging and noting that $\binom{1}{2} = 0$, we get

$$2\left(\binom{2}{2} + \cdots + \binom{k}{2}\right) + \left(\binom{1}{1} + \cdots + \binom{k}{1}\right).$$

The first part of this sum can be evaluated by applying (9) with $m + n = k$ and $k = 2$. The second part is evaluated by setting $m + n = k$ and $k = 1$. This gives

$$2\binom{k+1}{3} + \binom{k+1}{2}.$$

Expanding this gives the final answer:

$$1^2 + 2^2 + \cdots + k^2 = \frac{n(n+1)(2n+1)}{6}.$$

75. Representations of r^n in terms of binomials are not unique. The method of undetermined coefficients is a fool-proof way of getting one such representation, but there may be others. It so happens that

$$r^2 = \binom{r}{2} + \binom{r+1}{2}$$

is another representation for r^2. Use it to find the sum of

$$1^2 + 2^2 + 3^2 + \cdots + k^2$$

by the binomial method.

76. Use **71** to find the sum of

$$1^3 + 2^3 + 3^3 + \cdots + k^3.$$

by the binomial method.

77. A triangular number $t(k)$ is the sum of all integers from 1 to k. Here are the first few triangular numbers:

$$1,\ 3 = 1 + 2,\ 6 = 1 + 2 + 3,\ 10 = 1 + 2 + 3 + 4.$$

As shown in problem **73**, the kth triangular number is given by (12),

$$t(k) = \frac{k(k+1)}{2}.$$

Use the binomial method to evaluate the sum of the first k triangular numbers:

$$1 + 3 + 6 + 10 + \cdots + t(k).$$

78. Prove this beautiful identity:

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = (1 + 2 + 3 + \cdots + k)^2$$

using the results of problems **73** and **76**.

79. Pentagonal numbers are defined by

$$p(n) = \frac{k(3k-1)}{2}.$$

They are called pentagonal numbers because it is possible to arrange $p(k)$ dots in the form of a regular pentagon. Use the binomial method to evaluate the sum of the first k pentagonal numbers:

$$p(1) + p(2) + \cdots + p(k).$$

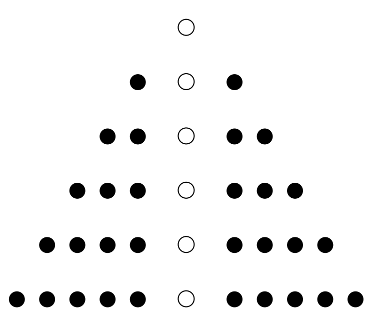
80. Prove the following identity for the sum of the fourth powers of the first k integers:

$$1^4 + 2^4 + \cdots + k^4 = \frac{k(k+1)(2k+1)(3k^2+3n-1)}{30}.$$

Use **72** and the binomial method.

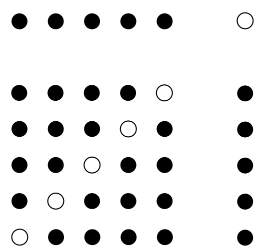
81. The sum of the first k positive odd numbers is k^2. Give a graphical proof.

Represent the odd numbers in the form $2k-1$. This way $k=1$ gives 1, which the first positive odd number. Odd numbers can be drawn by making a dot in the middle and then "wings" on both sides:



$2(1) - 1 = 1$
$2(2) - 1 = 3$
$2(3) - 1 = 5$
$2(4) - 1 = 7$
$2(5) - 1 = 9$
$2(6) - 1 = 11$

These figures can be folded and arranged into a perfect square with side k and area k^2.



82. Give a proof of **81** using the binomial method.

83. Use the result of problem **70** and the binomial method to prove this relationship for the sum of the squares of consecutive odd numbers:

$$1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2 = \frac{k(2k-1)(2k+1)}{3}.$$

84. Use this representation of r^2

$$r^2 = \binom{r+1}{2} + \binom{r}{2}$$

to sum the series in problem **83**.

85. Use the binomial method to prove the following identity involving the sum of the cubes of odd numbers:

$$1^3 + 3^3 + 5^3 + \cdots + (2k-1)^3 = k(3k^3 - 6k^2 - 2k + 6).$$

86. Evaluate this sum by using the binomial method:

$$1 \times 2 \,+\, 2 \times 3 \,+\, 3 \times 4 \,+ \cdots +\, (k-1)(k).$$

The first step is to express each term in the form of binomial numbers. Rearrange the terms like this:

$$2 \times 1 \,+\, 3 \times 2 \,+\, 4 \times 3 \,+ \cdots +\, k(k-1).$$

Notice that each term is just two times a binomial number:

$$2\binom{2}{2} + 2\binom{3}{2} + \cdots + 2\binom{k}{2}.$$

This is so because

$$r(r-1) = 2\binom{r}{2}.$$

The Pascal triangle diagonal identity (9) with $m + n = k$ and $m = 2$ can be immediately applied to sum the series:

$$2\binom{2}{2} + 2\binom{3}{2} + \cdots + 2\binom{k}{2} = 2\binom{k+1}{3}$$
$$= \frac{(k-1)(k)(k+1)}{3}.$$

87. Use the result of problem **86** to sum the series

$$1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + 9 \times 10.$$

88. Use the binomial method to find the sum of

$$1 \times 2 \times 3 + 2 \times 3 \times 4 + \cdots + (k-2)(k-1)(k).$$

89. Use your result from problem **88** to evaluate

$$1 \times 2 \times 3 + 2 \times 3 \times 4 + \cdots + 8 \times 9 \times 10.$$

90. Evaluate this series of odd products

$$1 \times 3 + 3 \times 5 + 5 \times 7 + \cdots + (2(k-1)-1)(2k-1)$$

using the binomial method.

91. Use the binomial method to sum this series:

$$1 \times 2 \times 3 \times 4 + 2 \times 3 \times 4 \times 5 + \cdots$$
$$\cdots + (k-3)(k-2)(k-1)(k).$$

We will use a slightly different approach for this problem. The terms of this series can be written as factorials, and these factorials can be written as binomial numbers:

$$1 \times 2 \times 3 \times 4 = \frac{4!}{0!} = 4!\frac{4!}{4!\,0!} = 4!\binom{4}{0}$$

$$2 \times 3 \times 4 \times 5 = \frac{5!}{1!} = 4!\frac{5!}{4!\,1!} = 4!\binom{5}{1}$$

$$3 \times 4 \times 5 \times 6 = \frac{6!}{2!} = 4!\frac{6!}{4!\,2!} = 4!\binom{6}{2}.$$

The last term is

$$\begin{aligned}(k-3)(k-2)(k-1)k &= \frac{k!}{(k-4)!} \\ &= 4!\frac{k!}{4!(k-4)!} \\ &= 4!\binom{k}{k-4}.\end{aligned}$$

The series we want to evaluate can now be written as

$$4!\left(\binom{4}{0}+\binom{5}{1}+\cdots+\binom{k}{k-4}\right).$$

Seting $m+n=k$ and $n=k+4$ in diagonal identity (11) immediately gives

$$4!\binom{k+1}{k-4}.$$

By the Pascal symmetry relationship, this is the same as

$$4!\binom{k+1}{5}.$$

We obtained the other equivalent form of this because we applied a different diagonal sum identity. The similarity to the result of problem **86** becomes even more clear if we expand this into a polynomial in k and then factor it:

$$\frac{(k-3)(k-2)(k-1)(k)(k+1)}{5}.$$

This bears a striking similarity to the final result of problem **86**, which suggests that both are special cases of something deeper.

92. Study the patterns in the solutions of problems **86**, **88** and **91**. Make a conjecture that covers all such sums, including the sum in problem **73**.