# An Application of the Error Function to Mechanics

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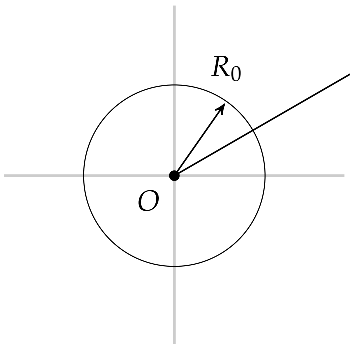
Consider a particle attracted to an infinitely long cylinder or linear mass concentrated at the origin. If the particle is released at a distance $R$, what is $T$, the time of fall?

On pages 76–78 of his classic book, *A Treatise on Dynamics of a Particle*[1], Peter Guthrie Tait gives solutions to this and other related problems of elementary mechanics. We will extend Tait's solution to the case where the particle is released from a distance $R$ but does not fall all the way to the origin. Rather, it falls to a fixed distance $R_0$ from the origin.




Can we determine $T$? Or, given $T$, can we determine the gravitational constant? We can, in terms of the error function.

The equation of motion of the particle is

$$m\frac{d^2r}{dt^2} = mg(r),$$

where $m$ is the mass of the particle, $r$ is its distance from the origin at time $t$ and $g(r)$ is the acceleration of gravity at point $r$. For an infinitely long cylinder or wire at the origin, we have

$$\frac{d^2r}{dt^2} = -\frac{\mu}{r}. \tag{1}$$

If we interpret $\mu$ as the universal gravitational constant $G$ times the linear mass density $\lambda$ of the infinite cylinder, the product

$$\frac{G\lambda}{r}$$

ends up having the dimensions of acceleration, as it should.

[1] Tait and Steele, *Treatise on Dynamics of a Particle*, 1900, MacMillan, London.

After multiplying the equation of motion (1) by the velocity $dr/dt$ we have:

$$\left(\frac{d^2r}{dt^2}\right)\frac{dr}{dt} = -\frac{\mu}{r}\frac{dr}{dt}.$$

Now both sides can be immediately integrated:

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = -\mu\log(r) + C. \tag{2}$$

Tait says that we can't integrate this any further, but he provides an interesting technique for finding $T$ from (2) by examining the boundary condition of the problem.

We release the particle at position $R$. It begins at rest. Therefore we have the boundary condition

$$\left.\frac{dr}{dt}\right|_{r=R} = 0.$$

This condition is satisfied if $C = \mu\log R$:

$$\begin{aligned}\frac{1}{2}\left(\frac{dr}{dt}\right)^2 &= -\mu\log(r) + \mu\log(R)\\ &= \mu\log\frac{R}{r}.\end{aligned} \tag{3}$$

Clearly, when $r = R$, the right hand side is zero, which implies that $dr/dt$ on the left is zero too.

Taking the roots of both sides of (3) gives

$$\frac{dr}{dt} = -\sqrt{2\mu\log\frac{R}{r}}.$$

We have taken the negative root because $r$ gets smaller as $t$ increases, so $dx/dt$ is negative. It's better to make the sign explicit. Separate the variables,

$$dt = \frac{-1}{\sqrt{2\mu\log\frac{R}{r}}}\,dr$$

and integrate both sides:

$$\int_0^T dt = \frac{1}{\sqrt{2\mu}}\int_R^{R_0}\frac{-1}{\sqrt{\log\frac{R}{r}}}\,dr. \tag{4}$$

Let's take a moment to explain the limits of integration on the left and right hand sides of (4). The particle begins its journey at time $t = 0$ and completes it at $t = T$. In terms of position $r$, this corresponds to a journey from $R$ to $R_0$. The left hand side of (4) can be immediately integrated, giving $T$. If we could integrate the right hand side, we would have our solution for the fall time. Following Tait's method, we make a change of variable by letting

$$u(r) = \sqrt{\log \frac{R}{r}}.$$

Under this transformation, the variable $r$, the differential $dr$ and the limits of integration $R$ and $R_0$ become:

$$\begin{aligned} r &= Re^{-u^2} \\ dr &= -2Re^{-u^2} u\, du \\ u(R) &= \sqrt{\log \frac{R}{R}} = 0 \\ u(R_0) &= \sqrt{\log \frac{R}{R_0}}. \end{aligned}$$

Using these transformations on (4), we get

$$T = \frac{1}{\sqrt{2\mu}} \int_0^{\sqrt{\log(R/R_0)}} \left(\frac{-1}{u}\right) (-2R) e^{-u^2} u\, du$$

or,

$$T = \frac{2R}{\sqrt{2\mu}} \int_0^{\sqrt{\log(R/R_0)}} e^{-u^2}\, du. \tag{5}$$

What seemed like a hopelessly complicated mess beginning with (4) now falls into place upon introducing the the error function:

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du.$$

Rewriting (5) in terms of erf, we have our expression for the time of fall $T$:

$$T = R\sqrt{\frac{\pi}{2\mu}}\ \operatorname{erf} \sqrt{\log \frac{R}{R_0}}.$$

Let's look at the problem in a different way, if only to arrange our solution into a more attractive form.

Suppose we could release a particle at a distance $R$ and measure the time it takes to get to $R_0$. If we had such data, could we determine the gravitational constant? In fact our last formula gives us what we need to do that. We just need to rearrange it a bit:

$$\mu = \frac{\pi R^2}{2T^2} \left(\operatorname{erf} \sqrt{\log \frac{R}{R_0}}\right)^2. \tag{6}$$

I hope you agree that (6) is a remarkably beautiful expression for $\mu$.

If $R_0$ is made smaller, we approach the case where the particle falls all the way to the origin, which was the problem solved by Tait. Let's recover Tait's solution by taking the limit:

$$R_0 \to 0, \quad \log \frac{R}{R_0} \to \infty.$$

But $\operatorname{erf} \infty$ is 1. So (6) becomes

$$\mu = \frac{\pi}{2} \frac{R^2}{T^2}$$

which is equivalent to the solution given by Tait.