The Anharmonic Group and Functional Equations

Ted Szylowiec

tedszy@gmail.com

We find ways of solving (and posing) a certain type of functional equation competition problem by exploiting its connection to geometry and the anharmonic group.

1 Proportional division

Proportional division of a line segment is a very useful technique to know, which can be applied to many types of geometry problems. It is such a fruitful concept that there exists an entertaining little book about it. [1].

Here we will apply the idea of proportional division to the discovery of a quantity that remains invariant under changes of perspective: the *anharmonic ratio*. The concept of proportional division leads very naturally into that of anharmonic ratio, with the added benefit of making the anharmonic ratio definition easy to remember and the proof of its invariance very elegant and straight-forward.

We are given two reference points A and B which are the endpoints of a segment, and we want to find a point M, given the proportion $AM : MB$. Notice that M does not have to

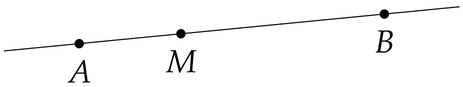

Fig. 1. Internal division.

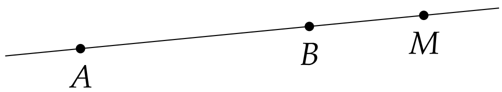

Fig. 2. External division.

be in between A and B. In Figure 1, M internally divides segment AB, while in Figure 2, point M externally divides AB. In the case of internal division, both AM and MB are positive, while in external division, one of AM or MB will be negative.

So, given A, B and $AM : MB$, how can we find M? We will use the method of vectors. Consider Figure 3. We can choose the origin for vectors to be anywhere, so place the origin at point O. Let λ and μ be given numbers so that the proportion specifying the location of point M is

$$AM : MB = \lambda : \mu.$$

Let $\mathbf{A}$ be the vector from O to A, and let $\mathbf{B}$ be the vector from O to B. We seek $\mathbf{M}$, the vector from O to M. We can reach point M by going from O to M directly, or by first going to A and then to M. In terms of vectors,

$$\mathbf{M} = \mathbf{A} + \mathbf{AM}$$

where $\mathbf{AM}$ is the vector from A to M. But this vector is parallel to the vector $\mathbf{AB}$, and we know that two vectors are parallel if and only if one is proportional to the other. It is easy to see that this proportionality factor must be $\lambda/(\lambda+\mu)$. Together with $\mathbf{AB} = \mathbf{B} - \mathbf{A}$ we

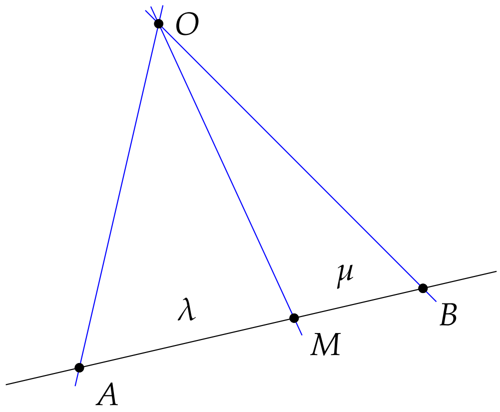

Fig. 3. Finding M.

have

$$\begin{aligned}\mathbf{M} &= \mathbf{A} + \mathbf{AM} \\ &= \mathbf{A} + \frac{\lambda}{\lambda+\mu}\mathbf{AB} \\ &= \frac{\mathbf{A}(\lambda+\mu) + \lambda(\mathbf{B}-\mathbf{A})}{\lambda+\mu}\end{aligned}$$

finally giving the beautiful proportional division formula

$$\mathbf{M} = \frac{\mu\mathbf{A} + \lambda\mathbf{B}}{\lambda+\mu}. \tag{1}$$

There is an occasion for error here. When writing this formula down quicky, it is tempting to automatically associate λ with $\mathbf{A}$ because that's how it looks in Figure 3, but that would be wrong. The order of λ and μ are reversed in the numerator of (1). Of course this follows from the derivation, but is there another way we can understand why the order is reversed?

Examine Figure 3. Consider what happens when λ becomes small. The segment AM shrinks until M coincides with A. Taking the limit as λ approaches zero in (1) should yield the same:

$$\lim_{\lambda\to 0} \frac{\mu\mathbf{A} + \lambda\mathbf{B}}{\lambda+\mu} = \frac{\mu\mathbf{A} + 0\mathbf{B}}{0+\mu} = \mathbf{A}.$$

As expected. Now imagine what happens when μ approaches 0. The segment MB gets smaller and smaller until M coincides with B.

$$\lim_{\mu\to 0} \frac{\mu\mathbf{A} + \lambda\mathbf{B}}{\lambda+\mu} = \frac{0\mathbf{A} + \lambda\mathbf{B}}{\lambda+0} = \mathbf{B}.$$

Again, as expected. This won't work if λ and μ are reversed in the numerator!

We don't need to know the values of λ and μ to find point M. We only need the ratio λ/μ. To see this, divide the numerator and denominator of (1) by μ:

$$\mathbf{M} = \frac{\mathbf{A} + \dfrac{\lambda}{\mu}\mathbf{B}}{1 + \dfrac{\lambda}{\mu}}.$$

The ratio λ/μ behaves like a coordinate. By choosing this ratio appropriately, we can place point M anywhere with respect to A and B. We can call λ/μ the *ratio* of M with respect to the base points A and B.

Any positive value for λ/μ gives a point M somewhere in between A and B. But when $\lambda/\mu = 1$, the point M coincides precisely with the midpoint of AB. When the ratio λ/μ is negative, the point M is *outside* of the segment AB. If we can visualise the line through A and B as actually being a circle of infinite radius, we get the following possibilities for the ratio of point M:

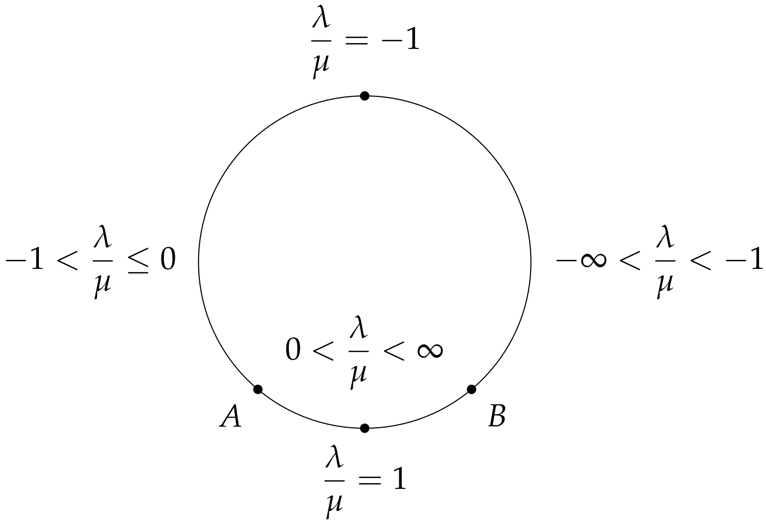

Fig. 4. The full range of ratio coordinates.

Interestingly, when $\lambda/\mu = -1$, M becomes the infinite point, which from the diagram is the external analogy of the internal midpoint of AB.

2 Change of perspective

To see how ratio coordinates change with changes in perspective, we will use a technique called *method of areas*. It is based on the following idea: if two triangles have the same vertex and share the same base line, then their bases are proportional to their areas. In Figure 5, triangle AMC shares part of the base line AB with triangle ABC. They also share vertex C.

Let's introduce a conventient notation for determinants. Write the columns of a determinant as a vector:

$$\begin{vmatrix}\mathbf{A} & \mathbf{B}\end{vmatrix} = \begin{vmatrix} A_x & B_x \\ A_y & B_y \end{vmatrix}.$$

Determinant properties are easily expressed in this notation. For example

$$\begin{vmatrix}\mathbf{A} & \mathbf{B}\end{vmatrix} = -\begin{vmatrix}\mathbf{B} & \mathbf{A}\end{vmatrix}$$

and

$$\begin{vmatrix}a\mathbf{A} & b\mathbf{B}\end{vmatrix} = ab\begin{vmatrix}\mathbf{A} & \mathbf{B}\end{vmatrix}$$

and also

$$\begin{vmatrix}a\mathbf{A} + b\mathbf{B} & \mathbf{C}\end{vmatrix} = a\begin{vmatrix}\mathbf{A} & \mathbf{C}\end{vmatrix} + b\begin{vmatrix}\mathbf{B} & \mathbf{C}\end{vmatrix}.$$

They can be verified by expanding out the determinants and showing that the left and right hand sides are the same.

If $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are the vectors to the vertices of triangle ABC, then the area of triangle ABC, denoted by the symbol (ABC), is given by the remarkable formula

$$(ABC) = \frac{1}{2}\left(\left|\mathbf{A} \quad \mathbf{B}\right| + \left|\mathbf{B} \quad \mathbf{C}\right| + \left|\mathbf{C} \quad \mathbf{A}\right|\right). \tag{2}$$

We can apply this to the method of areas illustrated in Figure 5. Let the origin O be at C and let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{M}$, $\mathbf{O}$ be vectors to points A, B M and O. The method of areas tells us, for

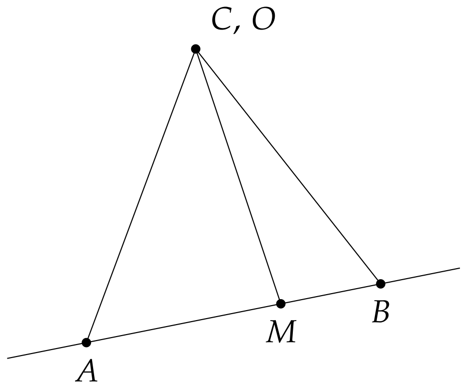

Fig. 5. Method of areas.

example, that segment AM is to AB as area (AMO) is to (ABO):

$$\frac{AM}{AB} = \frac{(AMO)}{(ABO)}.$$

The triangle areas can be calculated by (2). But there is a simplification, since $\mathbf{O}$ is the vector to the origin, then $\left|\mathbf{A} \quad \mathbf{O}\right|$ is zero, giving

$$\begin{aligned}(AMO) &= \frac{1}{2}\left(\left|\mathbf{A} \quad \mathbf{M}\right| + \left|\mathbf{M} \quad \mathbf{O}\right| + \left|\mathbf{O} \quad \mathbf{A}\right|\right) \\ &= \frac{1}{2}\left|\mathbf{A} \quad \mathbf{M}\right|\end{aligned}$$

and likewise $(ABO) = \frac{1}{2}\left|\mathbf{A} \quad \mathbf{B}\right|$. So we have, by the method of areas,

$$\frac{AM}{AB} = \frac{\left|\mathbf{A} \quad \mathbf{M}\right|}{\left|\mathbf{A} \quad \mathbf{B}\right|}.$$

This gives us a way to compute ratio coordinates *without having to compute the length of segments*, but instead by by means of determinants.

In Figure 6, O is the center of perspective. Lines from O project the base points A and B along with M onto a new set of collinear points A', B' and M' on some other line. What is the relationship between the ratio λ/μ of M and the ratio λ'/μ' of M'? Using the method of areas, and noting that determinants involving $\mathbf{O}$ are zero, we find λ/μ:

$$\frac{\lambda}{\mu} = \frac{AM}{MB} = \frac{(AMO)}{(MBO)} = \frac{\frac{1}{2}\left(\left|\mathbf{A} \quad \mathbf{M}\right| + \left|\mathbf{M} \quad \mathbf{O}\right| + \left|\mathbf{O} \quad \mathbf{A}\right|\right)}{\frac{1}{2}\left(\left|\mathbf{M} \quad \mathbf{B}\right| + \left|\mathbf{B} \quad \mathbf{O}\right| + \left|\mathbf{O} \quad \mathbf{M}\right|\right)} = \frac{\left|\mathbf{A} \quad \mathbf{M}\right|}{\left|\mathbf{M} \quad \mathbf{B}\right|}.$$

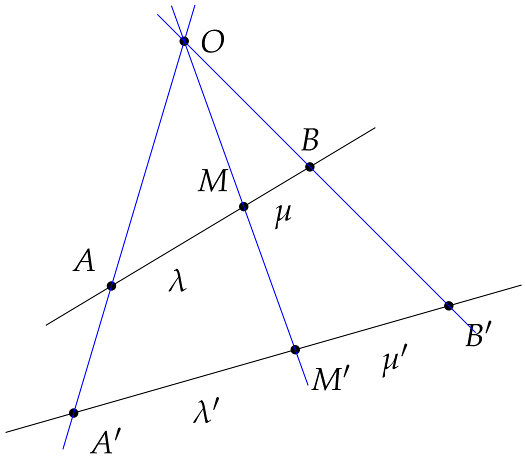

Fig. 6. Effect of change of perspective on ratio coordinates.

In the same way, using method of areas, we compute the ratio of the projected point M':

$$\frac{\lambda'}{\mu'} = \frac{A'M'}{M'B'} = \frac{(A'M'O)}{(M'B'O)} = \frac{\frac{1}{2}\left(\left|\mathbf{A}' \quad \mathbf{M}'\right| + \left|\mathbf{M}' \quad \mathbf{O}\right| + \left|\mathbf{O} \quad \mathbf{A}'\right|\right)}{\frac{1}{2}\left(\left|\mathbf{M}' \quad \mathbf{B}'\right| + \left|\mathbf{B}' \quad \mathbf{O}\right| + \left|\mathbf{O} \quad \mathbf{M}'\right|\right)} = \frac{\left|\mathbf{A}' \quad \mathbf{M}'\right|}{\left|\mathbf{M}' \quad \mathbf{B}'\right|}.$$

But vector $\mathbf{A}'$ is parallel to vector $\mathbf{A}$, vector $\mathbf{M}'$ is parallel to $\mathbf{M}$ and vector $\mathbf{B}'$ is parallel to $\mathbf{B}$. Therefore we have

$$\mathbf{A}' = a\mathbf{A} \quad \mathbf{B}' = b\mathbf{B} \quad \mathbf{M}' = m\mathbf{M}.$$

Putting these into the determinant expression for λ'/μ' gives

$$\frac{\lambda'}{\mu'} = \frac{\left|\mathbf{A}' \quad \mathbf{M}'\right|}{\left|\mathbf{M}' \quad \mathbf{B}'\right|} = \frac{\left|a\mathbf{A} \quad m\mathbf{M}\right|}{\left|m\mathbf{M} \quad b\mathbf{B}\right|} = \frac{am\left|\mathbf{A} \quad \mathbf{M}\right|}{mb\left|\mathbf{M} \quad \mathbf{B}\right|} = \frac{a\left|\mathbf{A} \quad \mathbf{M}\right|}{b\left|\mathbf{M} \quad \mathbf{B}\right|} = \frac{a\lambda}{b\mu}.$$

And so the ratio coordinate of point M is not invariant with change of perspective, but changes by a factor of a/b. This gives us the right idea for finding a quantity that does not change. If we take *two* ratios of two different points M_1 and M_2, and divide these, each will contribute a factor of a/b and they will cancel out. So while the ratio of M is not an invariant, it leads us to something which *is* an invariant: *the ratio of two ratios*. We call this quantity the *anharmonic ratio*.

In Figure 7 we have two base points A and B. Point M_1 has ratio λ_1/μ_1 and point M_2 has ratio λ_2/μ_2 with respect to the base points. Tranditionally, the anharmonic ratio of four points is denoted by the symbol (ABM_1M_2) and it is defined as

$$(ABM_1M_2) = \frac{\lambda_1/\mu_1}{\lambda_2/\mu_2} = \frac{\lambda_1\mu_2}{\mu_1\lambda_2}$$

Some books give defintions of anharmonic ratio in terms of segment lengths, with mnemonic rules to help you remember how it goes. But the approach above is fool-proof. You won't easily forget how anharmonic ratio is derived and defined.

Because we have already laid the ground-work, it is very easy to demonstrate that anharmonic ratio is invariant with change of perpective. Let the center of perspective be at O and let rays project points A, B, M_1 and M_2 onto another line, giving new points A', B', M_1' and M_2'. As usual, let $\mathbf{O}$, $\mathbf{A}$, $\mathbf{B}$, etc., be vectors to the points in the figure. Because $\mathbf{A}$

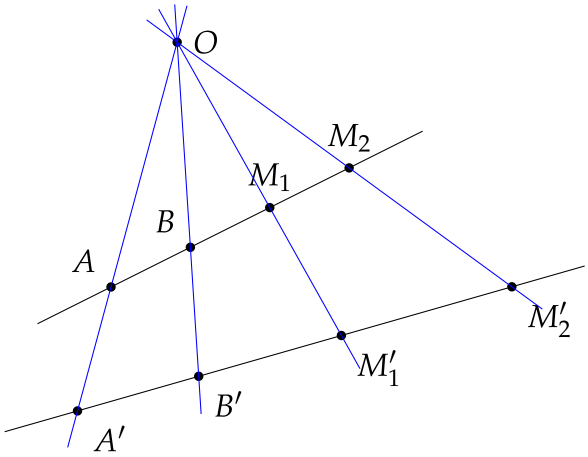

Fig. 7. Effect of change of perspective on anharmonic ratio.

is parallel to $\mathbf{A}'$, etc., we have

$$\mathbf{A}' = a\mathbf{A}, \quad \mathbf{B}' = b\mathbf{B}, \quad \mathbf{M}_1' = m\mathbf{M}_1, \quad \mathbf{M}_2' = n\mathbf{M}_2.$$

Since we know how the ratio changes, we can easily figure out the anharmonic ratio of the projected points.

$$(A'B'M_1'M_2') = \frac{\lambda_1'/\mu_1'}{\lambda_2'/\mu_2'} = \frac{\dfrac{|\mathbf{A}' \quad \mathbf{M}_1'|}{|\mathbf{M}_1' \quad \mathbf{B}'|}}{\dfrac{|\mathbf{A}' \quad \mathbf{M}_2'|}{|\mathbf{M}_2' \quad \mathbf{B}'|}} = \frac{\dfrac{|a\mathbf{A} \quad m\mathbf{M}_1|}{|m\mathbf{M}_1 \quad b\mathbf{B}|}}{\dfrac{|a\mathbf{A} \quad n\mathbf{M}_2|}{|n\mathbf{M}_2 \quad b\mathbf{B}|}} = \frac{\dfrac{am}{mb}\dfrac{|\mathbf{A} \quad \mathbf{M}_1|}{|\mathbf{M}_1 \quad \mathbf{B}|}}{\dfrac{an}{nb}\dfrac{|\mathbf{A} \quad \mathbf{M}_2|}{|\mathbf{M}_2 \quad \mathbf{B}|}} = \frac{\dfrac{|\mathbf{A} \quad \mathbf{M}_1|}{|\mathbf{M}_1 \quad \mathbf{B}|}}{\dfrac{|\mathbf{A} \quad \mathbf{M}_2|}{|\mathbf{M}_2 \quad \mathbf{B}|}} = \frac{\lambda_1/\mu_1}{\lambda_2/\mu_2}$$

And this is equal to (ABM_1M_2). Therefore if a range of four points is in perspective with another range of four points, both ranges have the same anharmonic ratio.

3 The anharmonic group

Is there anything special about the arrangement (ABM_1M_2)? What if we made M_1 and M_2 the base points? Or what if B and M_1 were made the base points? The results, (M_1M_2AB) and (BM_1AM_2) would also have the same anharmonic property with respect to changes of perspective. In fact, any permutation of the symbols in $(ABCD)$ will produce a quantity that is preserved with changes in perspective. There are $4! = 24$ ways to permute $(ABCD)$. How are these permutations related to each other?

We can study this question by choosing (ABM_1M_2) to be the fundamental anharmonic ratio. Call it x.

$$(ABM_1M_2) = \frac{\lambda_1/\mu_1}{\lambda_2/\mu_2} = \frac{AM_1}{M_1B}\frac{M_2B}{AM_2} = x.$$

The other permutations of the fundamental anharmonic ratio can be expressed in terms of this x. To make this job simpler, we introduce a new notation. Instead of the traditional symbol for anharmonic ratio $(ABCD)$, we will dispense with the parentheses and use sans-serif numbers. In what follows we will be using standard notation for permutations, cycles and transpositions. That uses parentheses in a similar way, so it is best to avoid ambiguity from the outset. Besides, it will be much easier to compute things with the new notation, as you will see. In the new notation, the fundamental anharmonic ratio is

$$x = 1234 = \frac{13}{32}\frac{42}{14}. \tag{3}$$

It is a daunting task to determine the effect of all 24 possible permutations on x. What is a logical approach to the problem? All permutations can be written as a sequence of transpositions, so we can begin our investigation by examining the six possible transpositions of 1234. Let's begin with the following three transpositions, giving them symbols α, β and γ.

$$\left(\begin{smallmatrix}1&2&3&4\\2&1&3&4\end{smallmatrix}\right) = (1\ 2) = \alpha$$
$$\left(\begin{smallmatrix}1&2&3&4\\4&2&3&1\end{smallmatrix}\right) = (1\ 4) = \beta$$
$$\left(\begin{smallmatrix}1&2&3&4\\3&2&1&4\end{smallmatrix}\right) = (1\ 3) = \gamma.$$

The full permutation is given first, then in cyclic notation. A cycle like $(a\ b\ c)$ means that a becomes b, b becomes c and then c becomes a. A 2-cycle $(a\ b)$ is then a transposition: a and b are swapped. It is a convenient notation and much easier to write than the full permutation in the form of a 2×4 array. Transpositions are applied from right to left. What does $\alpha = (1\ 2)$ do?

$$\alpha\,.\,1234 = (1\ 2)\,.\,1234 = 2134 = \frac{23}{31}\frac{41}{24}$$

Comparing this with (3) and noting that directed segments change signs when they are reversed, we have:

$$\alpha\,.\,1234 = \frac{23}{31}\frac{41}{24} = \frac{-32}{-13}\frac{-14}{-42} = \frac{32}{13}\frac{14}{42} = \frac{1}{x}.$$

Next is the effect of $\beta = (1\ 4)$.

$$\beta\,.\,1234 = (1\ 4)\,.\,1234 = 4231 = \frac{43}{32}\frac{12}{41}$$

It is not immediately obvious what $\beta\,.\,1234$ is. But with the help of Euler's identity for directed segments:

$$BC\,.\,AD + CD\,.\,AB + DB\,.\,AC = 0.$$

we can figure it out. Euler's identity is true for any four points A, B, C, D on a straight line. Let's write the identity in the new notation

$$23\,.\,14 + 34\,.\,12 + 42\,.\,13 = 0$$

and divide both sides by $32\,.\,14$:

$$\frac{23\,.\,14}{32\,.\,14} + \frac{34\,.\,12}{32\,.\,14} + \frac{42\,.\,13}{32\,.\,14} = 0$$

The middle term is $\beta\,.\,1234$ while the third term is, from (3) equal to x:

$$-1 + \beta\,.\,1234 + x = 0$$

or $\beta\,.\,1234 = 1 - x$. The effect of the γ transposition can be deduced in a similar way.

$$\gamma\,.\,1234 = (1\ 3)\,.\,1234 = 3214 = \frac{31}{12}\,\frac{42}{34}.$$

If we divide x by $1 - x$ we obtain

$$\frac{x}{1-x} = \frac{\frac{13}{32}\,\frac{42}{14}}{\frac{43}{32}\,\frac{12}{41}} = \frac{13}{32}\,\frac{42}{14} \times \frac{32}{43}\,\frac{41}{12} = -\frac{13}{43}\,\frac{42}{12} = -\gamma\,.\,1234.$$

In other words, $\gamma\,.\,1234 = x/(x-1)$. Arrange the effects of these transpositions in a table for easy reference.

$$\begin{array}{lll} \alpha = (1\ 2) & \alpha\,.\,1234 = \frac{23}{31}\,\frac{41}{24} & \alpha\,.\,x = \frac{1}{x} \\ \beta = (1\ 4) & \beta\,.\,1234 = \frac{43}{32}\,\frac{12}{41} & \beta\,.\,x = 1 - x \\ \gamma = (1\ 3) & \gamma\,.\,1234 = \frac{31}{12}\,\frac{42}{34} & \gamma\,.\,x = \frac{x}{x-1}. \end{array} \tag{4}$$

Applying the same transposition twice gives the identity. For example:

$$\alpha\alpha\,.\,1234 = \alpha\,.\,2134 = 1234.$$

And therefore, $\alpha\alpha = i$ and likewise $\beta\beta = i$ and $\gamma\gamma = i$. Now comes the job of computing products like $\alpha\beta$, $\beta\alpha$ and so on. It's much easier to determine these products by working with x rather than the range 1234. But we will do $\alpha\beta$ both ways to illustrate.

$$\alpha\beta\,.\,x = \alpha\,.\,(1-x) = 1 - \frac{1}{x} = \frac{x-1}{x}.$$

Doing it the long way and if comparing with (4) we see that

$$\alpha\beta\,.\,1234 = (1\ 2)(1\ 4)\,.\,1234 = (1\ 2)\,.\,4231 = 4132 = \frac{43}{31}\,\frac{21}{42} = \frac{x-1}{x}.$$

While $\beta\alpha$ is determined from

$$\beta\alpha\,.\,x = \beta\,.\,\frac{1}{x} = \frac{1}{1-x}$$

or, the long way,

$$\beta\alpha\,.\,1234 = (1\ 4)(1\ 2)\,.\,1234 = (1\ 4)\,.\,2134 = 2431 = \frac{23}{34}\,\frac{14}{21} = \frac{1}{1-x}$$

Nether $\alpha\beta$ nor $\beta\alpha$ is equivalent to any transposition in (4), so let us define $\alpha\beta = \tau$ and $\beta\alpha = \sigma$ and add two more entries to the table in (4):

$$\begin{array}{lll} \tau = \alpha\beta = (1\ 2)(1\ 4) & \tau\,.\,1234 = \frac{43}{31}\,\frac{21}{42} & \tau\,.\,x = \frac{x-1}{x} \\ \sigma = \beta\alpha = (1\ 4)(1\ 2) & \sigma\,.\,1234 = \frac{23}{34}\,\frac{14}{21} & \sigma\,.\,x = \frac{1}{1-x} \end{array} \tag{5}$$

Along with i, the identity element, the elements α, β, γ, σ and τ of (4) and (5) form a six element group called *the anharmonic group*. All other permutations of 1234 are equivalent to one of these elements. We will show this in the next part.

It is a straightforward but laborious task to construct a group multiplication table for the elements i, α, β, γ, σ and τ. The group operation is transformation of variable. The table is not symmetrical and the anharmonic group is non-abelian. We will give just one example of computing a table entry.

$$\sigma\gamma\,.\,x = \sigma\,.\,\frac{x}{x-1} = \frac{\dfrac{1}{1-x}}{\dfrac{1}{1-x}-1} = \frac{\dfrac{1}{1-x}}{\dfrac{x}{1-x}} = \frac{1}{x} = \alpha\,.\,x.$$

Therefore $\sigma\gamma = \alpha$. The multiplication table in Figure 8 is read left-to-right: the element in the leftmost column is on the left of the multiplication, so for example, $\beta\gamma = \tau$.

	i	α	β	γ	σ	τ
i	i	α	β	γ	σ	τ
α	α	i	τ	σ	γ	β
β	β	σ	i	τ	α	γ
γ	γ	τ	σ	i	β	α
σ	σ	β	γ	α	τ	i
τ	τ	γ	α	β	i	σ

Fig. 8. Anharmonic group multiplication table.

4 The elements of S_4

The group of permutations on four objects is called S_4. There are 24 ways to permute an arrangement of four things, such as the symbols 1234. So S_4 has 24 elements. But we claimed earlier that there are only six ways to permute 1234 such that they lead to distinct values of anharmonic ratio. What about the other 18 elements of S_4? Perhaps you're thinking that it will be a lot of work to determine what all 24 elements of S_4 do to the anharmonic ratio. But if we begin with the right approach, there is actually very little work involved. The approach is to write the elements of S_4 as transpositions and cycles and then connect them to elements of the anharmonic group. A remarkable symmetry is revealed.

Viewed as transpositions and cycles, there are four kinds of elements in S_4 (aside from the identity element):

2-cycles (transpositions):	$(1\ 2),\ (1\ 3),\ (1\ 4),\ (2\ 3),\ (2\ 4),\ (3\ 4)$
Products of disjoint transpositions:	$(1\ 2)(3\ 4),\ (1\ 3)(2\ 4),\ (1\ 4)(2\ 3)$
3-cycles:	$(1\ 2\ 3),\ (1\ 3\ 4),\ (1\ 3\ 2),\ (1\ 3\ 4),$ $(1\ 4\ 2),\ (1\ 4\ 3),\ (2\ 3\ 4),\ (2\ 4\ 3)$
4-cycles:	$(1\ 2\ 3\ 4),\ (1\ 2\ 4\ 3),\ (1\ 3\ 2\ 4),$ $(1\ 3\ 4\ 2),\ (1\ 4\ 2\ 3),\ (1\ 4\ 3\ 2).$

We already know $(1\ 2) = \alpha$, $(1\ 4) = \beta$ and $(1\ 3) = \gamma$. All we have to do is find the values of the other three transpositions. Once we have all the transpositions, the rest of S_4 can be computed very easily because any cycle can be written as a product of transpositions.

We have to compute the other three transpositions the long way, by examining their effects on 1234 and comparing with the table in (4).

$$(2\ 3)\,.\,1234 = 1324 = \frac{12}{23}\,\frac{43}{14} = \beta\,.\,1234$$
$$(2\ 4)\,.\,1234 = 1432 = \frac{13}{34}\,\frac{24}{12} = \gamma\,.\,1234$$
$$(3\ 4)\,.\,1234 = 1243 = \frac{14}{42}\,\frac{32}{13} = \alpha\,.\,1234$$

And so, to summarize all the transpositions,

$$\begin{array}{lll} (1\ 1) = \alpha & (1\ 3) = \gamma & (1\ 4) = \beta \\ (2\ 3) = \beta & (2\ 4) = \gamma & (3\ 4) = \alpha. \end{array} \tag{6}$$

The rest follows quickly. The products of disjoint cycles are equivalent to the identity:

$$\begin{aligned} (1\ 2)(3\ 4) &= \alpha\alpha = i \\ (1\ 3)(2\ 4) &= \gamma\gamma = i \\ (1\ 4)(2\ 3) &= \beta\beta = i. \end{aligned} \tag{7}$$

Any k-cycle can be written as the product of transpositions in the following way:

$$(a_1\ a_2 \ldots\ a_k) = (a_1\ a_k)(a_1\ a_{k-1}) \cdots (a_1\ a_2). \tag{8}$$

In case you want to do your own computations, let's do a complete example of how this works. Consider the 4-cycle $(1\ 2\ 3\ 4)$. Write it in full permutation array-form and note its effect on 1234:

$$(1\ 2\ 3\ 4)\,.\,1234 = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{smallmatrix}\right).\,1234 = 2341.$$

Using (8) we decompose $(1\ 2\ 3\ 4)$ into transpositions and obtain the same result:

$$(1\ 2\ 3\ 4)\,.\,1234 = (1\ 4)(1\ 3)(1\ 2)\,.\,1234 = (1\ 4)(1\ 3)\,.\,2134 = (1\ 4)\,.\,2314 = 2341.$$

If you want to learn more about permutations, transpositions and cycles, a good treatment is chapter 6 of Armstrong. [2]

Using (8) and (6), all the 3-cycles can be shown to be equivalent to either σ or τ, four instances of each:

$$\begin{aligned}
(1\,2\,3) &= (1\,3)(1\,2) = \gamma\alpha = \tau \\
(1\,2\,4) &= (1\,4)(1\,2) = \beta\alpha = \sigma \\
(1\,3\,2) &= (1\,2)(1\,3) = \alpha\gamma = \sigma \\
(1\,3\,4) &= (1\,4)(1\,3) = \beta\gamma = \tau \\
(1\,4\,2) &= (1\,2)(1\,4) = \alpha\beta = \tau \\
(1\,4\,3) &= (1\,3)(1\,4) = \gamma\beta = \sigma \\
(2\,3\,4) &= (2\,4)(2\,3) = \gamma\beta = \sigma \\
(2\,4\,3) &= (2\,3)(2\,4) = \beta\gamma = \tau
\end{aligned} \tag{9}$$

Again, with (8) and (6), the 4-cycles can all be shown to be either α, β or γ, two of each.

$$\begin{aligned}
(1\,2\,3\,4) &= (1\,4)(1\,3)(1\,2) = \beta\gamma\alpha = \beta\tau = \gamma \\
(1\,2\,4\,3) &= (1\,3)(1\,4)(1\,2) = \gamma\beta\alpha = \gamma\sigma = \beta \\
(1\,3\,2\,4) &= (1\,4)(1\,2)(1\,3) = \beta\alpha\gamma = \beta\sigma = \alpha \\
(1\,3\,4\,2) &= (1\,2)(1\,4)(1\,3) = \alpha\beta\gamma = \alpha\tau = \beta \\
(1\,4\,2\,3) &= (1\,3)(1\,2)(1\,4) = \gamma\alpha\beta = \gamma\tau = \alpha \\
(1\,4\,3\,2) &= (1\,2)(1\,3)(1\,4) = \alpha\gamma\beta = \alpha\sigma = \gamma
\end{aligned} \tag{10}$$

Considering the results of (6), (7), (9) and (10), together with the identity element, we see that each element in the anharmonic group corresponds to four equivalent elements in S_4.

5 Application to functional equations

We will look at three examples of competition-style functional equation problems. All three have something in common.

Our first functional equation is this:

$$f\left(\frac{1}{x}\right) + f\,(1-x) = x.$$

What can we say about it? Notice that the arguments of f have been transformed by elements of the anharmonic group! That is the common thread between all three of the functional equations that we will examine. Once we notice this, we can apply the properties of the anharmonic group to help us find solutions. Write the equation in a more general form using the notation developed earlier for elements of the anharmonic group and write f rather than $f(x)$:

$$\alpha\,.\,f + \beta\,.\,f = \phi. \tag{11}$$

Use the group multiplication table in Figure 8. Multiplying both sides of (11) by α gives:

$$\begin{aligned}
\alpha\alpha\,.\,f + \alpha\beta\,.\,f &= \alpha\,.\,\phi \\
f + \tau\,.\,f &= \alpha\,.\,\phi.
\end{aligned} \tag{12}$$

Now apply τ to both sides of (12):

$$\begin{aligned}
\tau\,.\,f + \tau\tau\,.\,f &= \tau\alpha\,.\,\phi \\
\tau\,.\,f + \sigma\,.\,f &= \gamma\,.\,\phi.
\end{aligned} \tag{13}$$

Apply β to the original equation (11):

$$\begin{aligned} \beta\alpha \,.\, f + \beta\beta \,.\, f &= \beta \,.\, \phi \\ \sigma \,.\, f + f &= \beta \,.\, \phi. \end{aligned} \tag{14}$$

Now, adding equations (12) and (14) gives:

$$2f + \tau \,.\, f + \sigma \,.\, f = (\alpha + \beta) \,.\, \phi.$$

Use (13) to eliminate $\tau \,.\, f + \sigma \,.\, f$:

$$2f + \gamma \,.\, \phi = (\alpha + \beta) \,.\, \phi$$

from which we have the following general solution for f:

$$f = \frac{1}{2}(\alpha + \beta - \gamma) \,.\, \phi. \tag{15}$$

Putting $\phi = x$ and $\alpha \,.\, x = 1/x$, $\beta \,.\, x = 1 - x$ and $\gamma \,.\, x = x/(x-1)$ and doing some algebra, we obtain the particular solution:

$$f(x) = \frac{x^3 - x^2 + 1}{2x(1-x)}.$$

You can verify this solution by plugging it into the original functional equation.

Our next problem is to find a solution to

$$f(x) + f\left(\frac{x-1}{x}\right) = x + \frac{1}{x}.$$

We will dispense with writing the little dot ".". Just keep in mind that applications of the elements of the anharmonic group are not literal multiplications, but transformations of the argument of $f(x)$. As before, we write the functional equation with a more general right-hand side:

$$f + \tau f = \phi. \tag{16}$$

Apply τ to both sides of (16):

$$\begin{aligned} \tau f + \tau\tau f &= \tau\phi \\ \tau f + \sigma f &= \tau\phi. \end{aligned} \tag{17}$$

And now apply σ to (16):

$$\begin{aligned} \sigma f + \sigma\tau f &= \sigma\phi \\ \sigma f + f &= \sigma\phi. \end{aligned} \tag{18}$$

Adding (16) and (18) gives

$$2f + \tau f + \sigma f = \phi + \sigma\phi$$

and now $\tau f + \sigma f$ can be eliminated by means of (17),

$$2f + \tau f = (i + \sigma)f$$

from which we get a general solution

$$f = \frac{1}{2}(i + \sigma - \tau)\phi. \tag{19}$$

Let's take a slightly different approach to get a solution to the original problem. Notice that the right-hand side of (16) can be written in terms of anharmonic group transformations:

$$x + \frac{1}{x} = i \,.\, x + \alpha \,.\, x = (i + \alpha) \,.\, x$$

The solution (19) becomes:

$$f = \frac{1}{2}(i + \sigma - \tau)(i + \alpha) \,.\, x = \frac{1}{2}(i + \alpha + \sigma + \beta - \tau - \gamma) \,.\, x = \frac{x^2 - x + 2}{2x(1 - x)}.$$

Our final example is a more complicated anharmonic-type functional equation:

$$f\left(\frac{1}{x}\right) + 2f\,(1 - x) + 3f\left(\frac{x}{x - 1}\right) = x \tag{20}$$

As before, write this in terms of anharmonic transformations and ϕ:

$$\alpha f + 2\beta f + 3\gamma f = \phi. \tag{21}$$

If we operate on this with σ and τ, we get two new equations:

$$\begin{aligned} 3\alpha f + \beta f + 2\gamma f &= \sigma\phi \\ 2\alpha f + 3\beta f + \gamma f &= \tau\phi \end{aligned} \tag{22}$$

Equation (21) and the two equations in (22) form a system of three linear equations in three unknowns αf, βf, γf. Solving for these unknowns with, say, Cramer's method, we have:

$$\begin{aligned} \alpha f &= (\tau\phi + 7\sigma\phi - 5\phi)/18 \\ \beta f &= (7\tau\phi - 5\sigma\phi + \phi)/18 \\ \gamma f &= (-5\tau\phi + \sigma\phi + 7\phi)/18 \end{aligned}$$

Since $\alpha^2 = i$ it is easy to recover f from the first solution:

$$f = \frac{1}{18}(\beta\phi + 7\gamma\phi - 5\alpha\phi).$$

For $\phi = x$, as in the original problem (20) we obtain

$$f(x) = \frac{x^3 - 9x^2 + 6x - 5}{18x(1 - x)}.$$

The expressions for βf and γf give the same $f(x)$.

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